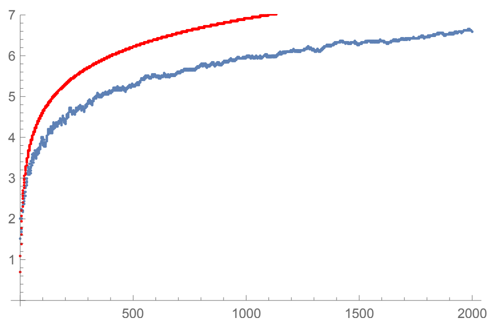
Outline of the Proof of the Prime Number Theorem

Definition: Let n be a natural number. Then $\pi(n)$ is the number of prime numbers $\leq n$. A graph of $\frac{n}{\pi(n)}$ suggests that $\frac{n}{\pi(n)} \sim \log n$:



Prime Number Theorem:

$$\lim_{n \to \infty} \frac{\pi(n)\log(n)}{n} = 1.$$

Analytic Continuation: Let $f : S \to \mathbb{C}$ be holomorphic on S. If $S \subseteq T$ and $F : T \to \mathbb{C}$ is holomorphic on T and satisfies $F(z) = f(z)$ for all $z \in S$, then we say that F is an analytic continuation of f to the set T.

Riemann Zeta Function: For all $z \in \mathbb{C}$ with re$z > 1$,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

It is holomorphic on its domain.

The Euler Product Formula: For all $z \in \mathbb{C}$ with re $z > 1$,

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^z}}.$$

The Logarithmic Derivative of $\zeta(z)$: For all $z \in \mathbb{C}$ with re $z > 1$,

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\log p_n}{1 - p_n^z}.$$

Analytic Continuation of $\zeta(z)$ to $\{x \in \mathbb{C} : \text{re z} > 0\} - \{1\}$:

$$\zeta_1(z) = \frac{1}{z-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z} \right) \, dx.$$

Theorem: For all $z \neq 1$ with re $z = 1$, $\zeta_1(z) \neq 0$.

Tchebychev Theta Function:

$$\theta(x) = \sum_{p \leq x} \log p,$$

the sum ranging over prime numbers bounded above by x.

Theorem:

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1 \implies \lim_{n \to \infty} \frac{\pi(n) \log(n)}{n} = 1.$$

Theorem: If the improper integral

$$\int_0^{\infty} \theta(e^t) e^{-t} - 1 \; dt$$

converges then $\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$.

Theorem: For all $z \in \mathbb{C}$ with re $z > 0$,

$$\int_0^{\infty} (\theta(e^t) e^{-t} - 1) e^{-tz} \; dt = \frac{1}{z+1} \sum_{n=1}^{\infty} \frac{\log p_n}{p_n^{z+1}} - \frac{1}{z}.$$

Theorem: An analytic continuation of $\frac{1}{z+1} \sum_{n=1}^{\infty} \frac{\log p_n}{p_n^{z+1}} - \frac{1}{z}$ to all $z \in \mathbb{C}$ with re $z \geq 0$ is $I(z)$, the Laurent series expansion of

$$\frac{1}{z+1} \left(-\frac{\zeta_1'(z+1)}{\zeta_1(z+1)} - \sum_{n=1}^{\infty} \frac{\log p_n}{p_n^{2z+2} - p_n^{z+1}} \right) - \frac{1}{z}$$

about $z = 0$, which has no negative powers of z.

Theorem: $\lim_{T \to \infty} \int_0^T \theta(e^t) e^{-t} - 1 \; dt = I(0)$.