Complex Analysis Notes

Princeton Lectures In Analysis II

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<u>**The Field $\mathbb{C}$**</u>

Definition: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Addition:

$$(a + bi) + (a' + b'i) = (a + a') + (b + b')i.$$

This is associative and commutative.

$\mathbb{C}$ is a group under addition, with identity element $0+0i$ and inverse operation

$$-(a + bi) = (-a) + (-b)i.$$

Multiplication:

$$(a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i.$$

This is associative and commutative:

$\mathbb{C}^*$ is a group under multiplication, with identity element $1 + 0i$ and inverse operation

$$(a + bi)^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i.$$

One check that multiplication is distributive. Hence $\mathbb{C}$ is a field.

Complex Conjugation:

$$\overline{a + bi} = a - bi.$$

One can check that complex conjugation is a field isomorphism, i.e. that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$ for all $z, w \in \mathbb{C}$.

Norm:

$$||a + bi|| = \sqrt{a^2 + b^2},$$

$$z\overline{z} = ||z||^2.$$

Real and Imaginary Parts:

$$\text{re }(z) = \frac{z+\overline{z}}{2}, \qquad \text{im}(z) = \frac{z-\overline{z}}{2i}$$

Lemma: re $(z) \leq ||z||$.

Proof: This follows from $a \leq \sqrt{a^2+b^2}$. $\square$

Triangle Inequality: For all $z, w \in \mathbb{C}$, $||z+w|| \leq ||z|| + ||w||$.

Proof:

$$||z+w||^2 = (z+w)(\overline{z}+\overline{w}) = ||z||^2 + z\overline{w} + w\overline{z} + ||w||^2 =$$

$$||z||^2 + 2\text{re }(z\overline{w}) + ||w||^2 \leq ||z||^2 + 2||z\overline{w}|| + ||w||^2 =$$

$$||z||^2 + 2||z||||w|| + ||w||^2 = (||z|| + ||w||)^2.$$

$\square$

Corollary: For all $z, w \in \mathbb{C}$, $|||z|| - ||w||| \leq ||z-w||$.

Proof: This follows from $||z|| \leq ||z-w|| + ||w||$ and $||w|| \leq ||w-z|| + ||z||$. $\square$

Euler's Notation: For $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Trigonometric identities yield

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

Polar Form: Every $z \in \mathbb{C}$ lies on a circle of radius $r \geq 0$ about the origin and can be expressed in the form $z = re^{i\theta}$ where $r > 0$ and $\theta \in \mathbb{R}$. In fact, $r = ||z||$ and θ is any angle satisfying $r\cos\theta = \text{re } z$ and $r\sin\theta = \text{im}z$. Using Euler's notation we can see that complex multiplication can be interpreted in terms of rotation and dilation.

Solutions to $z^n = c$ where $c \neq 0$: Write $c = re^{i\theta}$ where $r > 0$. We seek all $z = se^{i\psi}$ with $s > 0$ satisfying

$$s^n e^{in\psi} = re^{i\theta}.$$

We must have $s = r^{\frac{1}{n}}$ and $e^{in\psi} = e^{i\theta}$. This forces $n\psi = \theta + 2k\pi$ where $k \in \mathbb{Z}$, or $\psi = \frac{\theta}{n} + \frac{2k}{n}\pi$, which yields

$$z = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k}{n}\pi\right)}.$$

There are n distinct values of z, corresponding to $0 \leq k < n$.

Example: The three complex solutions to $z^3 = 1$ are

$$\frac{-1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{-1}{2} - \frac{\sqrt{3}}{2}i,\ 1.$$

Sequences in $\mathbb{C}$

Definition: $\lim_{n\to\infty} z_n = z$ if and only if

$$\forall \epsilon > 0 : \exists N : n \geq N \implies ||z_n - z|| < \epsilon.$$

Example: $\lim_{n\to\infty} \left(\frac{1+n}{2n} + \frac{2n^5}{3n^5-1000}i\right) = \frac{1}{2} + \frac{2}{3}i.$

Proof: Let $\epsilon > 0$ be given. We wish to find N so that $n > N$ implies $||\left(\frac{1+n}{2n} + \frac{2n^5}{3n^5-1000}i\right) - \left(\frac{1}{2} + \frac{2}{3}i\right)|| < \epsilon$, or equivalently $||\frac{1}{2n} + \frac{2000}{9n^5-3000}i|| < \epsilon$. Given that

$$\left|\left|\frac{1}{2n} + \frac{2000}{9n^5-3000}i\right|\right| \leq \left|\left|\frac{1}{2n}\right|\right| + \left|\left|\frac{2000}{9n^5-3000}i\right|\right| = \left|\frac{1}{2n}\right| + \left|\frac{2000}{9n^5-3000}\right|,$$

it suffices to require $\frac{1}{2n} < \frac{\epsilon}{2}$ and $\frac{2000}{9n^5-3000} < \frac{\epsilon}{2}$. The first inequality occurs when $n > \frac{2}{2\epsilon}$. Given that $9n^5 - 3000 > 8n^5$ when, for examle, $n > 10$, we have

$$\frac{2000}{9n^5-3000} < \frac{2000}{8n^5} \leq \frac{2000}{8n} < \frac{\epsilon}{2}$$

when $n > \frac{4000}{8\epsilon}$. So we can choose any N greater than all three of the numbers $\frac{2}{2\epsilon}, 10, \frac{4000}{8\epsilon}$. $\square$

Example: Let $z \in \mathbb{R}$ satisfying $0 < ||z|| < 1$. Then $\lim_{n\to\infty} z^n = 0$.

Proof: Let $\epsilon > 0$ be given. We wish to find N so that $n > N$ implies $||z^n|| < \epsilon$, or equivalently $\left(\frac{1}{||z||}\right)^n > \frac{1}{\epsilon}$. Write $\frac{1}{||z||} = 1 + \theta$ where $\theta > 0$. By the Binomial Theorem, $\left(\frac{1}{||z||}\right)^n = (1+\theta)^n \geq 1 + n\theta$. We wish to require $1 + n\theta > \frac{1}{\epsilon}$. We just need any natural N satisfying $N > \frac{\frac{1}{\epsilon}-1}{\theta} = \frac{\frac{1}{\epsilon}-1}{\frac{1}{||z||}-1}$. $\square$

Theorem: A convergent sequence cannot have two distinct limits.

Proof: Suppose $z_n \to w$ and $z_n \to w'$ where $w \neq w'$. Then for each n we have $||w - w'|| \leq ||w - z_n|| + ||z_n - w'||$, and for sufficiently large n, $||z_n - w|| < \frac{||w-w'||}{2}$ and $||z_n - w'|| < \frac{||w-w'||}{2}$, which implies $||w - w'|| < ||w - w'||$, a contradiction. $\square$

Theorem: Assume (z_n) converges to z. Then every subsequence (z_{n_k}) converges to z.

Proof: Let $\epsilon > 0$ be given. Then there exists N such that $k \geq N$ implies $||z_k - z|| < \epsilon$, hence $k \geq N$ implies $n_k \geq k \geq N$ implies $||z_{n_k} - z|| < \epsilon$. $\square$

Theorem: Assume that (z_n) is convergent and that a subsequence (z_{n_k}) converges to z. Then (z_n) converges to z.

Proof: If (z_n) converges to w then (z_{n_k}) converges to w. By uniqueness of limits, $w = z$. Hence (z_n) converges to z. $\square$

Theorem: If $z_n \to z$ then $||z_n|| \to ||z||$.

Proof: This follows from $|||z_n|| - ||z||| \leq ||z_n - z|| \to 0$. $\square$

The Sum, Product, and Quotient Rules

Theorem: Assume $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Then:

(1) $\lim_{n\to\infty} z_n + w_n = z + w$

(2) $\lim_{n\to\infty} z_n w_n = zw$

(3) When $w_n \neq 0$ for all n and $w \neq 0$, $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$.

Proof:

(1) We have

$$||(z_n + w_n) - (z + w)|| \leq ||z_n - z|| + ||w_n - w||.$$

In order to make this quantity $< \epsilon$, it suffices to make $||z_n - z|| < \frac{\epsilon}{2}$ and $||w_n - w|| < \frac{\epsilon}{2}$. Given $\epsilon > 0$, we will choose N so that $n \geq N$ forces both inequalities.

(2) We have

$$||z_n w_n - zw|| \le ||z_n w_n - zw|| + ||zw - zw_n|| = ||z_n - z|| ||w|| + ||w_n - w|| ||z||.$$

In order to make this quantity $< \epsilon$, it suffices to make $||z - z_n|| < \frac{\epsilon}{2(1+||w||)}$ and $||w - w_n|| < \frac{\epsilon}{2(1+||z||)}$. Given $\epsilon > 0$, we will choose N so that $n \ge N$ forces both inequalities.

(3) We have

$$\left|\left|\frac{z_n}{w_n} - \frac{z}{w}\right|\right| = \left|\left|\frac{z_n w - z w_n}{w w_n}\right|\right| \le \frac{||z_n - z||\,||w||}{||w||\,||w_n||} + \frac{||w - w_n||\,||z||}{||w||\,||w_n||}.$$

We will first show that the denominator contribution can be bounded above. Since $w_n \to w$ and $||w|| > 0$, there exists N_1 such that $n \ge N_1$ implies $|||w_n|| - ||w||| \le ||w_n - w|| < \frac{||w||}{2}$, which implies $||w_n|| > \frac{||w||}{2}$. Hence $n \ge N$ implies

$$\frac{1}{||w||\,||w_n||} \le \frac{2}{||w||^2}.$$

Now let $\epsilon > 0$ be given. Then there exists N_2 such that $n \ge N_2$ implies $||z - z_n|| ||w|| < \frac{\epsilon ||w||^2}{4}$ and $||w_n - w|| ||z|| < \frac{\epsilon ||w||^2}{4}$. Hence for any n larger than both N_1 and N_2,

$$\left|\left|\frac{z_n}{w_n} - \frac{z}{w}\right|\right| < \epsilon.$$

□

A Brief Review of the Topology of $\mathbb{R}$

Least Upper Bound Axiom: Every $S \subseteq \mathbb{R}$ that has an upper bound has a least upper bound.

Example: The set $(-\infty, 1)$ has many upper bounds, including the number 1. None of the numbers in $(-\infty, 1)$ is an upper bound, because if $t \in (-\infty, 1)$ then $\frac{t+1}{2} \in (-\infty, 1)$ as well, and since $t < \frac{t+1}{2}$, t cannot be an upper bound. Therefore 1 is the least upper bound of $(-\infty, 1)$.

Example: Fix $\sigma > 1$. Let $S = \{s_n : n \in \mathbb{N}\}$ where

$$s_n = \sum_{k=1}^{n} \frac{1}{k^\sigma} = \frac{1}{1^\sigma} + \frac{1}{2^\sigma} + \cdots + \frac{1}{n^\sigma}.$$

The set S is bounded above: for any $p \in \mathbb{N}$ we have

$$s_{2^p-1} = \sum_{i=1}^{p} \left(\sum_{k=2^{i-1}}^{2^i-1} \frac{1}{k^\sigma} \right) \le \sum_{i=1}^{p} \left(\sum_{k=2^{i-1}}^{2^i-1} \frac{1}{(2^{i-1})^\sigma} \right) = \sum_{i=1}^{p} \left(\frac{1}{2^{\sigma-1}} \right)^{i-1} =$$

$$\frac{1 - \left(\frac{1}{2^{\sigma-1}}\right)^p}{1 - \frac{1}{2^\sigma}} \le \frac{1}{1 - \frac{1}{2^\sigma}}.$$

For any $n \in \mathbb{N}$, $n \ge 2^p - 1$ for some $p \in \mathbb{N}$, hence $s_n \le s_{2^p-1} \le \frac{1}{1-\frac{1}{2^\sigma}}$ for all $n \in \mathbb{N}$. So S has a least upper bound.

Theorem: Let $a_1 \le a_2 \le a_3 \le \cdots$ be a bounded sequence of real numbers. Then (a_n) is convergent, and

$$\lim_{n\to\infty} a_n = a$$

where a is the least upper bound of $\{a_n : n \in \mathbb{N}\}$.

Proof: Let $\epsilon > 0$ be given. Since $a-\epsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$, there exists a natural number N such that $a_N > a - \epsilon$. For $n \ge N$ we have

$$a - \epsilon < a_N \le a_n \le a < a + \epsilon,$$

hence

$$|a_n - a| < \epsilon.$$

□

Example: Fix $\sigma > 1$. Let $s_n = \sum_{k=1}^{n} \frac{1}{k^\sigma}$. Then $s_1 < s_2 < \cdots$ is a bounded sequence of real numbers. Let s be the least upper bound of this sequence. Then

$$\sum_{k=1}^{\infty} \frac{1}{k^\sigma} = \lim_{n\to\infty} s_n = s.$$

Bolzano-Weierstrass Theorem: Let $M > 0$ be given. Then every sequence in $[-M, M]$ has a convergent monotonic subsequence in $[-M, M]$.

Proof: Let $(a_n) \subseteq [-M, M]$ be an arbitrary sequence of real numbers. If there is a strictly decreasing subsequence $a_{n_1} > a_{n_2} > a_{n_3} \cdots$, then the sequence $(-a_{n_k})$ is increasing and bounded, hence converges to a limit $a \in [-M, M]$ by the previous theorem. Therefore $\lim_{n\to\infty} a_{n_k} = -a \in [-M, M]$.

Now suppose that (a_n) does not have a strictly decreasing sequence. Then there must be a minimum number a_{n_1}. The sequence $a_{n_1+1}, a_{n_1+2}, \dots$ cannot have a strictly decreasing sequence, so there must be a minimum number a_{n_2}. The sequence $a_{n_2+1}, a_{n_2+2}, \dots$ cannot have a strictly decreasing sequence, so there must be a minimum number a_{n_3}. Keep on going. Then the subsequence $a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$ converges to a number $a \in [-M, M]$. $\square$

Real and Complex Cauchy Sequences

Definition: A sequence of real numbers (a_n) is Cauchy if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > m \geq N \implies |a_n - a_m| < \epsilon.$$

Theorem: A real sequence converges if and only if it is Cauchy.

Proof: Suppose (a_n) converges to a. Given $\epsilon > 0$, there exists N such that $n \geq N$ implies $|a_n - a| < \frac{\epsilon}{2}$, hence $n > m \geq N$ implies $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \epsilon$. Hence (a_n) is Cauchy.

Conversely, assume that (a_n) is Cauchy. Then it is bounded, since there exists N such that $n > N \implies |a_n - a_N| < 1$. Let (a_{n_k}) be a monotonic subsequence of (a_n). Then (a_{n_k}) converges to a limit a. This implies that (a_n) converges to a: let $\epsilon > 0$ be given. Then there exists N_1 such that $n > m \geq N_1$ implies $|a_n - a_m| < \frac{\epsilon}{2}$, and there exists N_2 such that $k \geq N_2$ implies $|a_{n_k} - a| < \frac{\epsilon}{2}$, hence $k > N_1, N_2$ implies

$$|a_k - a| \leq |a_k - a_{n_{N_2}}| + |a_{n_{N_2}} - a| < \epsilon.$$

$\square$

Definition: A sequence of complex numbers (z_n) is Cauchy if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > m \geq N \implies ||z_n - z_m|| < \epsilon.$$

Theorem: A complex sequence converges if and only if it is Cauchy.

Proof: Suppose (z_n) converges to z. Given $\epsilon > 0$, there exists N such that $n \geq N$ implies $||z_n - z|| < \frac{\epsilon}{2}$, hence $n > m \geq N$ implies $||z_n - a_z|| \leq ||z_n - z|| + ||z_m - z|| < \epsilon$. Hence (z_n) is Cauchy.

Conversely, suppose (z_n) is Cauchy. If $z_n = a_n + b_n i$ for each n, then (a_n) and (b_n) are both Cauchy because $|a_n - a_m| \leq ||z_m - z_n||$ and $|b_n - b_m| \leq ||z_n - z_m||$. Hence (a_n) converges to a limit a and (b_n) converges to a limit b, which implies that (z_n) converges to $a + bi$. $\square$

Topology of $\mathbb{C}$

Definition: A set $S \subseteq \mathbb{C}$ is bounded by M if $||z|| \leq M$ for all $z \in S$. Geometrically, all the points in S lie within the circle of radius M about the origin.

Definition: A set $S \subseteq \mathbb{C}$ is closed if and only if every convergent sequence in S has its limit in S.

Example: Consider the set $S = \{z \in \mathbb{C} : ||z|| \geq 1\}$. Suppose $(z_n) \subseteq S$ and $z_n \to z$. If $z \notin S$ then $||z|| < 1$, and there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $||z_n - z|| < 1 - ||z||$, which implies $|||z_N|| - ||z||| < ||z_N - z|| < 1 - ||z||$, which implies $||z_N|| < 1$, a contradiction. Therefore $z \in S$. Hence S is closed.

Definition: A set $S \subseteq \mathbb{C}$ is open if and only if for each $z \in S$ there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq S$, where

$$B_\epsilon(z) = \{w \in \mathbb{C} : ||w - z|| < \epsilon\}.$$

Example: Consider the set $S = \{z \in \mathbb{C} : ||z|| > 1\}$. Given $z \in S$, we claim that $B_{1-||z||}(z) \subseteq S$. To prove this, we have

$$w \in B_{1-||z||}(z) \implies ||w|| \leq ||w - z|| + ||z|| < 1 - ||z|| + ||z|| = 1 \implies w \in S.$$

Theorem: A set $S \subseteq \mathbb{C}$ is closed if and only if S^c is open.

Proof: Assume S is closed. If S^c is not open, then there exists $z \in S^c$ such that for each $n \in \mathbb{N}$ there exists $z_n \in B_{\frac{1}{n}}(z) \cap S$, which yields a sequence $(z_n) \subseteq S$ converging to $z \notin S$, a contradiction. Therefore S^c is open.

Conversely, Assume S^c is open. Let $(z_n) \subseteq S$ be convergent sequence with limit z. If $z \notin S$ then there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq S^c$. Since $z_n \to z$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $||z_n - z|| < \epsilon$, which implies $z_N \in B_\epsilon(z) \subseteq S^c$, a contradiction. Therefore $z \in S$. Hence S is closed. $\square$

Compact Subsets of $\mathbb{C}$

Definition: A set $S \subseteq \mathbb{C}$ is compact if and only if every sequence in S has a subsequence converging to a limit in S.

Example: Let $M > 0$ and $S = \{z \in \mathbb{C} : ||z|| \leq M\}$. If $(a_n + b_n i)$ is a sequence in S then (a_n) is a sequence in $[-M, M]$, hence a subsequence $(a_n : n \in I)$ converges to some $a \in [-M, M]$ by the Bolzano-Weierstrass Theorem. The sequence $(b_n : n \in I)$ is another sequence in $[-M, M]$, and a subsequence $(b_n : n \in J)$ converges some $b \in [-M, M]$, where $J \subseteq I$. Hence $(a_n + b_n i : n \in J)$ converges to $a + bi$. Since $||a_n + b_n i|| \leq M$ for each n, $||a + bi|| \leq M$, hence $a + bi \in S$. Hence S is compact.

Theorem: A set $S \subseteq \mathbb{C}$ is compact if and only if it is closed and bounded.

Proof: Assume S is compact. Then it must be bounded, otherwise S would contain a sequence of the form (z_n) where $||z_n|| > n$ for each n, and no subsequence of (z_n) converges. To show that S is closed, let $(z_n) \subseteq S$ be a convergent sequence. By compactness, a subsequence of (z_n) converges to a point $z \in S$, which implies that (z_n) converges to $z \in S$.

Conversely, assume that S is closed and bounded. Then there exists $M > 0$ such that $||z|| \leq M$ for all $z \in S$. Let (z_n) be an arbitrary sequence in S. By the example above, (z_n) has a subsequence (z_{n_k}) that converges to a point z in $\{z \in \mathbb{C} : ||z|| \leq M\}$. Since S is closed, $z \in S$. Hence S is compact. $\square$

Definition: Let $X \subseteq \mathbb{C}$ be a compact set. The diameter of X is

$$\operatorname{diam}(X) = \sup\{||x - y|| : x, y \in X\}.$$

Theorem: Let (X_n) be a sequence of non-empty compact sets satisfying

$$X_1 \supseteq X_2 \supseteq \cdots$$

and

$$\lim_{n \to \infty} \operatorname{diam}(X_n) = 0.$$

Then:

(1) $\bigcap_{n \in \mathbb{N}} X_n$ consists of a single point x_0.

(2) For any sequence (x_n) where $x_n \in X_n$ for each n, $x_n \to x_0$.

Proof: Let (x_n) be an arbitrary sequence satisfying $x_n \in X_n$ for each n. Then (x_n) is a Cauchy sequence: Let $\epsilon > 0$ be given. Then we can choose N so that diam $(X_N) < \epsilon$. When $n > m \geq N$, x_n and x_m belong to X_N, hence $||x_n - x_m|| \leq$ diam $(X_N) < \epsilon$. Therefore (x_n) converges to a limit x. Since the subsequence $(x_n, x_{n+1}, \dots)$ resides in X_n and converges to x, $x \in X_n$. Therefore $x \in \bigcap_{n \in \mathbb{N}} X_n$. If y is any other point in $\bigcap_{n \in \mathbb{N}} X_n$ then $||x - y|| \leq$ diam (X_n) for each n, hence $||x - y|| = 0$, hence $x = y$. Hence both (1) and (2) must be true. □

Compact Sets, Open Covers, and Lebesgue Numbers

Open Cover: Let S be a subset of $\mathbb{C}$. We say that $\{U_i : i \in I\}$ is an open cover of S if each U_i is open and $S \subseteq \bigcup_{i \in I} U_i$.

Example: Let $S = \{x + iy \in \mathbb{C} : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. An open cover of S is $\{B_{\frac{1}{100}}(z) : z \in S\}$.

Definition: Let $S \subseteq \mathbb{C}$ be a set and let $\mathcal{U}$ be an open cover of S. If $\epsilon > 0$ has the property that $B_\epsilon(z)$ is a subset of some $U \in \mathcal{U}$ for each $z \in S$, then ϵ is called a Lebesgue number of $\mathcal{U}$ with respect to S.

Theorem: Let $S \subseteq \mathbb{C}$ be a compact set and let $\mathcal{U}$ be an open cover of S. Then $\mathcal{U}$ has a Lebesgue number with respect to S.

Proof: Let $i \in \mathbb{N}$ be given. If $\frac{1}{i}$ is not a Lebesgue number then we can find $z_i \in S$ such that $B_{\frac{1}{i}}(z_i)$ is not a subset of any $U \in \mathcal{U}$. Now suppose that for each $i \in \mathbb{N}$, $\frac{1}{i}$ is not a Lebesgue number. By compactness of S, the sequence (z_i) must have a subsequence (z_{n_i}) that converges to a point $z \in S$. We have $z \in U_0$ for some $U_0 \in \mathcal{U}$. For each $i \in \mathbb{N}$, $B_{\frac{1}{n_i}}(z_{n_i}) \not\subseteq U_0$, so we can find $w_{n_i} \in B_{\frac{1}{n_i}}(z_{n_i})$ such that $w_{n_i} \notin U_0$. We have $z_{n_i} \to z$, hence $||z_{n_i} - z|| \to 0$. We also have $||w_{n_i} - z_{n_i}|| \to 0$. Hence $||w_{n_i} - z|| \leq ||w_{n_i} - z_{n_i}|| + ||z_{n_i} - z|| \to 0$, hence $w_{n_i} \to z \in U_0$. This is impossible since (w_{n_i}) is a convergent sequence in the closed set $\mathbb{C} - U_0$ and so must converge to a point in $\mathbb{C} - U_0$. So for some $i \in \mathbb{N}$, $\frac{1}{i}$ is a Lebesgue number.

Complex Functions and Continuity

A complex function is a mapping $f : S \to \mathbb{C}$ where $S \subseteq \mathbb{C}$. We will say that f is continuous at $z_0 \in S$ if and only if for all for all sequences (z_n) in S

$$\lim_{n \to \infty} z_n = z_0 \implies \lim_{n \to \infty} f(z_n) = f(z_0).$$

We will also say that f is continuous on S if and only if it is continuous at each $z \in S$.

Example: Using the sum and product rule it is easy to show that polynomial functions $f : \mathbb{C} \to \mathbb{C}$ of the form $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ are continuous on $\mathbb{C}$.

Example: Let $f(z)$ and $g(z)$ be polynomial functions, and assume that $g(z) \neq 0$ for all $z \in S$. Using the quotient rule combined with continuity of polynomial functions, the function $q : S \to \mathbb{C}$ defined by $g(z) = \frac{f(z)}{g(z)}$ is continuous on S.

Theorem: A function $f : S \to \mathbb{C}$ is continuous at $z \in S$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $w \in S$, $||w - z|| < \epsilon$ implies $||f(w) - f(z)|| < \epsilon$.

Proof: Assume that f is continuous at z. If the $\epsilon - \delta$ condition were false, then there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there would have to exist a $w_n \in S$ such that $||w_n - z|| < \frac{1}{n}$ and $||f(w_n) - f(z)|| \geq \epsilon$. Hence $\lim_{n\to\infty} w_n = z$ yet $\lim_{n\to\infty} f(w_n) \neq f(z)$, a contradiction. So the $\epsilon - \delta$ condition must be true.

Conversely, if the $\epsilon - \delta$ condition is true, let $z_n \to z$ in S. We will show that $f(z_n) \to f(z)$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $w \in S$ and $||w - z|| < \delta$ implies $||f(w) - f(z)|| < \epsilon$. Since $z_n \to z$, there exists N such that $n \geq N$ implies $||z_n - z|| < \delta$, which implies $||f(z_n) - f(z)|| < \epsilon$. $\square$

Theorem: Let $S \subseteq \mathbb{C}$ be a compact set and let $f : S \to \mathbb{C}$ be continuous on S. Then $f(S)$ is compact.

Proof: Let $(f(z_k))$ be a sequence in $f(S)$. Then (z_k) is a sequence in S, hence there must be a convergent subsequence (z_{n_k}) which has a limit $z \in S$. Since $z_{n_k} \to z$ and f is continous, $f(z_{n_k}) \to f(z)$. $\square$

Holomorphic Complex Functions

A complex function $f : S \to \mathbb{C}$ is said to be holomorphic at $z_0 \in S$ if and only if z_0 is an interior point of S and there exists a complex number w such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = w.$$

If w exists then we write $f'(z_0) = w$.

The precise definition of the limit above is

$$\forall \epsilon > 0 : \exists \delta > 0 : 0 < ||z - z_0|| < \delta \text{ and } z \in S \implies \left|\left|\frac{f(z) - f(z_0)}{z - z_0} - w\right|\right| < \epsilon.$$

Equivalent Definitions of $f'(z)$:

(1)

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} \to f'(z_0)$$

for all sequences $(z_n) \subseteq S$ satisfying $z_n \neq z_0$ and $z_n \to z_0$.

(2) The function $\Delta_{f,z_0} : S \to \mathbb{C}$ defined by

$$\Delta_{f,z_0}(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

is continuous at z_0.

Example: Let n be a positive integer and let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^n$. Then for any $z_0 \in \mathbb{C}$ we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1}) = nz_0^{n-1}.$$

Example: Let $g : \mathbb{C} - \{0\} \to \mathbb{C}$ be defined by $g(z) = \frac{1}{z}$. Then for any $z_0 \in \mathbb{C} - \{0\}$ we have

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{-1}{zz_0} = -\frac{1}{z_0^2}.$$

Theorem: If $f : S \to \mathbb{C}$ is holomorphic at z_0 then f is continuous at z_0.

Proof: We have

$$f(z) = f(z_0) + (z - z_0)\Delta_{f,z_0}(z)$$

for all $z \in S$. Let (z_n) be a sequence in S satisfying $z_n \to z_0$. By Equivalent Definition (2) of differentiability we have

$$f(z) \to f(z_0) + 0 \cdot f'(z_0).$$

$\square$

The Sum, Product, and Chain Rule for Complex Differentiation

Theorem: Let $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ be holomorphic at $z_0 \in \mathbb{C}$. Then $f + g : S \to \mathbb{C}$ and $fg : S \to \mathbb{C}$ are holomorphic at z_0 and we have

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

and

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Proof: The sum rule is a consequence of Equivalent Definition (1) of differentiability. To prove the product rule, observe that

$$\frac{f(z_n)g(z_n) - f(z_0)g(z_0)}{z - z_0} = \frac{f(z_n) - f(z_0)}{z - z_0} g(z_n) + f(z_0)\frac{g(z_n) - g(z_0)}{z_n - z_0}.$$

When $z_n \to z_0$ we have $g(z_n) \to g(z_0)$, hence

$$\frac{f(z_n)g(z_n) - f(z_0)g(z_0)}{z - z_0} \to f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

$\square$

Theorem: Let $g : S \to \mathbb{C}$ be holomorphic at z_0, let T be a subset of $\mathbb{C}$ containing $g(S)$, and let $f : T \to \mathbb{C}$ be holomorphic at $g(z_0)$. Then $f \circ g : S \to \mathbb{C}$ is holomorphic at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proof: Let $z_n \to z_0$ in S. Then $g(z_n) \to g(z_0)$ in $g(S)$, hence

$$\Delta_{f,g(z_0)}(g(z_n)) \to f'(g(z_0)),$$

$$\Delta_{g,z_0}(z_n) \to g'(z_0),$$

$$\Delta_{f \circ g, z_0}(z_n) = \Delta_{f,g(z_0)}(g(z_n)) \cdot \Delta_{g,z_0}(z_n) \to f'(g(z_0)) \cdot g'(z_0).$$

$\square$

Example: Let $f : S \to \mathbb{C}$ be holomorphic at z_0 and let $g : S \to \mathbb{C}$ be holomorphic at z_0 and non-zero on S. We can express the mapping $h : S \to \mathbb{C}$ defined by $h(z) = \frac{f(z)}{g(z)}$ in the form

$$h = f \cdot (r \circ g),$$

where $r : \mathbb{C} - \{0\} \to \mathbb{C}$ is defined by $r(z) = \frac{1}{z}$. Given that $r'(z) = -\frac{1}{z^2}$, the product and chain rules yield

$$h'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Some Real Analysis

Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ is compact.

Proof: Let $(f(x_n))$ be a sequence in $f([a, b])$. Then (x_n) is a sequence in $[a, b]$, and by the Bolzano-Weierstrass Theorem there is a convergent subsequence (x_{n_k}) with a limit x which must belong to $[a, b]$ by closure of $[a, b]$. By continuity of f, $(f(x_{n_k}))$ converges to $f(x) \in f([a, b])$. $\square$

Extreme Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ is bounded and there exists $c \in [a, b]$ such that the least upper bound of $f([a, b])$ is $f(c)$.

Proof: Since $f([a, b])$ is compact, it is bounded. Let y be the least upper bound of $f([a, b])$. Then for each n there exists $f(x_n) \in f([a, b])$ such that $y - \frac{1}{n} < f(x_n) \leq y$, hence $(f(x_n))$ converges to y. Since $f([a, b])$ is compact, it is closed, hence $y \in f([a, b])$. Hence $y = f(c)$ for some $c \in [a, b]$. $\square$

Mean Value Theorem: Assume $a < b$. Let $f : [a, b] \to \mathbb{R}$ be differentiable at each $x \in (a, b)$. Then

$$f(b) - f(a) = f'(c)(b - a)$$

for some $c \in (a, b)$.

Proof: Let $h : [a, b] \to \mathbb{R}$ be the function defined by

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then h is differentiable on $[a, b]$, and it suffices to prove that $h'(c) = 0$ for some $c \in (a, b)$. We have $h(a) = h(b)$, and we will assume without loss of generality that $h(a)$ is not the maximum output of h along $[a, b]$. Since h is continuous on $[a, b]$, by the Extreme Value Theorem there exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $c \in [a, b]$, and clearly $a < c < b$. Therefore

$$h'(c) = \lim_{n \to \infty} \frac{h(c + \frac{1}{n}) - h(c)}{\frac{1}{n}} \leq 0$$

and

$$h'(c) = \lim_{n \to \infty} \frac{h(c - \frac{1}{n}) - h(c)}{\frac{-1}{n}} \geq 0,$$

therefore $h'(c) = 0$. $\square$

Intermediate Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then for each k between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = k$.

Proof: We will assume without loss of generality that $f(a) < k < f(b)$. Suppose that $f(c) \neq k$ for all $k \in [a, b]$. Then the set

$$A = \{x \in [a, b] : f(x) < k\}$$

is closed: if $(x_k) \subseteq A$ converges to a point x then $x \in [a, b]$, hence by continuity $f(x_k) \to f(x)$, and since $f(x_k) < k$ for all k, $f(x) < k$, hence $x \in A$. Since A is closed and bounded, it is compact. Let $a_0 \in A$ be the least upper bound of A. Then for all natural numbers $n \geq \frac{1}{b - a_0}$, $a + \frac{1}{n} \in [a, b] - A$, hence $f(a_0 + \frac{1}{n}) > k$, hence

$$f(a_0) = \lim_{n \to \infty} f(a_0 + \frac{1}{n}) > k,$$

a contradiction. Therefore $f(c) = k$ for some $c \in [a, b]$. $\square$

Corollary: Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. If $f(a) < f(b)$ then f is strictly increasing on $[a, b]$, and if $f(a) > f(b)$ then f is strictly decreasing on $[a, b]$.

Proof: Assume $f(a) < f(b)$. If there exist $x_1 < x_2$ in $[a, b]$ such that $f(x_1) > f(x_2)$, then we must have $f(a) > f(x_2)$, otherwise $f(x_1) > f(x_2) > f(a)$ implies $f(x_2) = f(t)$ for some $t \in [a, x_1]$ by the Intermediate Value Theorem, which is impossible given that $x_2 \neq t$. Given $f(x_2) < f(a) < f(b)$, we must have $f(a) = f(t)$ for some $t \in [x_2, b]$ by the Intermediate Value Theorem, which is impossible given that $a \neq t$. , Therefore no such x_1 and x_2 exist, hence f strictly increases along $[a, b]$. The other case is similar. $\square$

Inverse Function Theorem: Let $a < b$ and let $f : [a, b] \to [c, d]$ be a bijective function.

(i) If f is continuous on $[a, b]$ then f^{-1} is continuous on $[c, d]$.

(ii) If f is differentiable on (a, b) then f^{-1} is differentiable on (c, d).

Proof: (1) We will assume without loss of generality that f is increasing on $[a, b]$. The Intermediate Value Theorem implies that for each $a' < b'$ in $[a, b]$, $f([a', b']) = [f(a'), f(b')]$.

Let $c < y < d$ and $\epsilon > 0$ be given. Write $f^{-1}(y) = x$. Then $[x - \epsilon_1, x + \epsilon_1] \subseteq [a, b]$ for some $0 < \epsilon_1 \leq \epsilon$, and $f([x - \epsilon_1, x + \epsilon_1]) = [y - \delta_1, y + \delta_2]$ for some $\delta_1, \delta_2 > 0$. Hence $f^{-1}([y-\delta_1, y+\delta_2]) = [x-\epsilon_1, x+\epsilon_2]$. Setting $\delta = \min(\delta_1, \delta_2)$, we have $f^{-1}((y - \delta, y + \delta)) \subseteq (x - \epsilon, y + \epsilon)$. In other words, $|t - y| < \delta$ implies $|f^{-1}(t) - f^{-1}(y)| < \epsilon$. Hence f^{-1} is continuous at y.

f^{-1} is continuous at c: Let $\epsilon > 0$ be given. Choose $0 < \epsilon_1 \leq \epsilon$ such that $[a, a + \epsilon_1] \subseteq [a, b]$. Then $f([a, a + \epsilon_1]) = [c, c + \delta]$ for some $\delta > 0$, hence $f^{-1}([c, c + \delta]) = [a, a + \epsilon_1]$. This implies $t \in [c, d]$ and $|t - c| < \delta$ implies $|f^{-1}(t) - f^{-1}(c)| < \epsilon$. f^{-1} is continuous at d by a similar argument.

(2) Let $y \to y_0$ in (c, d). Since f is differentiable on (a, b), it is continuous on (a, b), therefore f^{-1} is continuous on (c, d), therefore $f^{-1}(y) \to f^{-1}(y_0)$. Moreover, since f is strictly monotonic on $[a, b]$, the Mean Value Theorem implies that $f'(x) \neq 0$ for all $x \in (a, b)$. Write $f^{-1}(y) = x$ and $f^{-1}(y_0) = x_0$. Then we have

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} \to \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Hence f^{-1} is differentiable at y_0. $\square$

Complex Extreme Value Theorem

Let $S \subseteq \mathbb{C}$ be a compact set and let $f : S \to \mathbb{C}$ be continuous on S. Then

$$\sup\{||f(z)|| : z \in S\} = ||f(z_0)||$$

for some $z_0 \in S$.

Proof: It will suffice to show that the set $X = \{||f(z)|| : z \in S\}$ is compact, for then the least upper bound of X will be an element of X.

Let $(||f(z_n)||)$ be an arbitrary sequence in X. Then (z_n) is a sequence in S, and by compactness of S there must be a subsequence (z_{n_k}) converging to a point z_* in S. By continuity of f, $f(z_{n_k}) \to f(z_*)$, hence $||f(z_{n_k}) - f(z_*)|| \to 0$, hence

$$|||f(z_{n_k})|| - ||f(z_*)||| \leq ||f(z_{n_k}) - f(z_*)|| \to 0,$$

hence $||f(z_{n_k})|| \to ||f(z_*)||$. Since every sequence in X has a subsequence converging to a limit in X, X is compact. $\square$

The Cauchy-Riemann Equations

Let $f : S \to \mathbb{C}$ be holomorphic at $z_0 = a_0 + b_0 i$. For any sequence $(t_n) \subseteq \mathbb{R} - \{0\}$ satisfying $t_n \to 0$ we have

$$f'(z_0) = \lim_{n\to\infty} \frac{f(z_0 + t_n) - f(z_0)}{t_n}$$

and

$$f'(z_0) = \lim_{n\to\infty} \frac{f(z_0 + t_n i) - f(z_0)}{t_n i}.$$

If we write $f(x + iy) = u(x, y) + v(x, y)i$ for all $x + yi \in \mathbb{C}$, then these two equations imply

$$f'(z_0) = \lim_{n\to\infty} \frac{u(a_0 + t_n, b_0) - u(a_0, b_0)}{t_n} + \frac{v(a_0 + t_n, b_0) - v(a_0, b_0)}{t_n} i =$$

$$u_x(a_0, b_0) + v_x(a_0, b_0)i$$

and

$$f'(z_0) = \lim_{n\to\infty} \frac{u(a_0, b_0 + t_n) - u(a_0, b_0)}{t_n i} + \frac{v(a_0, b_0 + t_n) - v(a_0, b_0)}{t_n i} i =$$

$$-iu_y(a_0, b_0) + v_y(a_0, b_0).$$

Comparing the two expressions for $f'(z_0)$, we obtain

$$u_x(a_0, b_0) = v_y(a_0, b_0)$$

and

$$u_y(a_0, b_0) = -v_x(a_0, b_0).$$

These are called the Cauchy-Riemann Equations.

Example: Let $f(z) = z^2$. Then f is holomorphic on $\mathbb{C}$. We have $f(x+iy) = (x + iy)^2 = x^2 + 2xyi - y^2$, hence $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, and we can see that $u_x(a,b) = 2a = v_y(a,b)$ and $u_y(a,b) = -2b = -v_x(a,b)$ for all $a, b \in \mathbb{R}$.

Example: Let $f(z) = \overline{z}$. Then $f(x+iy) = x - iy$, hence $u(x,y) = x$ and $v(x,y) = -y$. Since $u_x(a,b) = 1$ and $v_y(a,b) = -1$ for all $a, b \in \mathbb{R}$, the Cauchy-Riemann equations do not hold at any $a+bi \in \mathbb{C}$, hence f is nowhere holomorphic.

Example: Satisfaction of the Cauchy-Riemann equations is necessary but not sufficient for differentiability: Let $f(x+iy) = x^{\frac{1}{3}}y^{\frac{2}{3}} + 0i$. Then f is identically 0 along the real and imaginary axes, hence

$$u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0,$$

so the Cauchy-Riemann equations are satisfied at $0+0i$. If $f'(0+0i)$ exists then for all $m \in \mathbb{R}$ we have

$$f'(0+0i) = \lim_{t \to 0} \frac{f(t+imt) - f(0)}{t+imt} = \frac{m^{\frac{2}{3}}}{1+im},$$

which is impossible. Hence f is not holomorphic at $0+0i$.

Theorem: Assume that $f : S \to \mathbb{C}$ satisfies the Cauchy-Riemann equations at $a+bi$, that $B_\epsilon(a+bi) \subseteq S$, and that u_x, u_y, v_x, v_y exist and are continuous on $B_\epsilon(a+bi)$. Then f is holomorphic at $a+bi$ and

$$f'(a+bi) = u_x(a,b) + v_x(a,b)i.$$

Proof: For any $(r,s) \in \mathbb{R}^2$ satisfying $||r+si|| < \epsilon$ we have

$$u(a+r, b+s) - u(a,s) =$$

$$u(a+r, b+s) - u(a, b+s) + u(a, b+s) - u(a,b) =$$

$$u_x(a_r, b+s)r + u_y(a, b_s)s$$

for some a_r between a and $a+r$ and some b_s between b and $b+s$ by the Mean Value Theorem. By continuity of u_x and u_y on $B_\epsilon(a+bi)$, we can write

$$u_x(a_r, b+s) = u_x(a,b) + \psi_1(r,s)$$

where $\psi_1(r,s) \to 0$ as $r+si \to 0+0i$. Similarly, we can write

$$u_y(a,b_s) = u_y(a,b) + \psi_2(r,s)$$

where $\psi_2(r,s) \to 0$ as $r+si \to 0+0i$. This yields

$$u(a+r,b+s) - u(a,s) = u_x(a,b)r + u_y(a,b)s + \psi_1(r,s)r + \psi_2(r,s)s.$$

Similarly, we have

$$v(a+r,b+s) - v(a,s) = v_x(a,b)r + v_y(a,b)s + \psi_3(r,s)r + \psi_4(r,s)s.$$

Suppressing some of the notation, and applying the Cauchy-Riemann equations, this yields

$$f((a+bi)+(r+si)) - f(a+bi) =$$

$$u_x r + u_y s + v_x ri + v_y si + (\psi_1+\psi_3 i)r + (\psi_2+\psi_4 i)s =$$

$$u_x r - v_x s + v_x ri + u_x si + (\psi_1+\psi_3 i)r + (\psi_2+\psi_4 i)s =$$

$$(r+si)\left(u_x + v_x i + (\psi_1+\psi_3 i)\frac{r}{r+si} + (\psi_2+\psi_4 i)\frac{s}{r+is}\right),$$

hence

$$\frac{f((a+bi)+(r+si)) - f(a+bi)}{r+si} = u_x + v_x i + (\psi_1+\psi_3 i)\frac{r}{r+si} + (\psi_2+\psi_4 i)\frac{s}{r+is}.$$

Since

$$\left\|\frac{r}{r+si}\right\| \le 1$$

and

$$\left\|\frac{s}{r+si}\right\| \le 1$$

and

$$\psi_1 + \psi_3 i \to 0+0i$$

and

$$\psi_2 + \psi_4 i \to 0+0i$$

as $r+si \to 0+0i$,

$$f'(a+bi) = u_x(a,b) + v_x(a,b)i.$$

$\square$

Complex Antiderivatives

Let $S \subseteq \mathbb{C}$ and let $f : S \to \mathbb{C}$ be holomorphic on S. We say that $F : S \to \mathbb{C}$ is an antiderivative of f on S if and only if F is holomorphic on S and $F'(z) = f(z)$ for all $z \in S$.

Example: $f(z) = z^2$, $F(z) = \frac{z^3}{3}$, $S = \mathbb{C}$.

Example: Let $S \subseteq \mathbb{C}$ be arbitrary, and let $f : S \to \mathbb{C}$ be defined by $f(x + iy) = x$. Suppose that $F : S \to \mathbb{C}$ is an antiderivative of f on S. Then by definition, each point of S is interior to S, and we must have $F(x + iy) = u(x, y) + v(x, y)i$ where the partial derivatives of u and v are continuous and satisfy the Cauchy-Riemann equations on S. Since

$$F'(x + iy) = u_x(x, y) + v_x(x, y)$$

for all $x + iy \in S$, we must have $u_x(x, y) = x$ and $v_x(x, y) = 0$ for all $x + iy \in S$. This implies that $u(x, y) = \frac{x^2}{2} + C(y)$ and $v(x, y) = D(y)$. The Cauchy Riemann equations force $x = D'(y)$ for all $x + iy \in S$, so each $y \in \mathbb{R}$ there is at most one $x \in \mathbb{R}$ such that $x + iy \in S$. This contradicts the fact that each point in S must be interior to S. Therefore f cannot have an antiderivative on S.

The Complex Exponential Function

We will define

$$e^{x+iy} = e^x e^{yi} = e^x(\cos y + i \sin y)$$

for all $x + iy \in \mathbb{C}$. One can check that the partial derivatives of $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are continuous and satisfy the Cauchy-Riemann equations on $\mathbb{C}$, hence e^z is holomorphic on $\mathbb{C}$. Since $u_x(x, y) = u(x, y)$ and $v_x(x, y) = v(x, y)$ for all x and y, $(e^z)' = e^z$ for all $z \in \mathbb{C}$. Moreover, if $z = x + iy$ and $z' = x' + iy'$, then

$$e^{z+z'} = e^{x+x'} e^{(y+y')i} = e^x e^{yi} \cdot e^{x'} e^{y'i} = e^z \cdot e^{z'}.$$

The complex exponential function is an extension of the real-valued exponential function to the complex plane.

Complex Trigonometric Functions

We will define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for all $z \in \mathbb{C}$. The following identities hold:

$$\sin'(z) = \cos(z),$$

$$\cos'(z) = -\sin(z),$$
$$\sin^2(z) + \cos^2(z) = 1,$$
$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w),$$
$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w).$$

A useful inequality is

$$||\sin(x+iy)||^2 = \frac{e^{2y} + e^{-2y} - 2\cos(2x)}{4} \geq \frac{e^{2y} + e^{-2y} - 2}{4} = \frac{e^{2|y|} + e^{-2|y|} - 2}{4} =$$

$$\left(\frac{e^{|y|} - e^{-|y|}}{2}\right)^2 \geq \frac{e^{2|y|}}{16} \geq \frac{1}{16}$$

for all $x, y \in \mathbb{R}$ satisfying $|y| \geq 1$, because

$$e^t - e^{-t} = e^t(1 - e^{-2t}) \geq e^t(1 - e^{-2}) \geq \frac{e^t}{2}$$

for all $t \geq 1$.

The Complex Logarithm

Let $S = \{x + iy \in \mathbb{C} : x > 0\}$. Then for all $a + bi \in S$, $B_a(a+bi) \subseteq S$. Define $u : (0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ and $v : (0, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ by

$$u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$$

and

$$v(x, y) = \tan^{-1}\left(\frac{y}{x}\right).$$

Then for any $(x, y) \in (0, \infty) \times \mathbb{R}$ we have

$$u_x = \frac{x}{x^2 + y^2},$$

$$u_y = \frac{y}{x^2 + y^2},$$

$$v_x = \frac{-y}{x^2 + y^2},$$

$$v_y = \frac{x}{x^2 + y^2}.$$

Thefore the function

$$f : S \to \{x + iy \in \mathbb{C} : (x, y) \in (-\infty, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})\}$$

defined by

$$f(x + iy) = \frac{1}{2} \ln(x^2 + y^2) + \tan^{-1} \left(\frac{y}{x}\right) i$$

is holomorphic at each $a + bi \in S$. Note that for $z = x + iy$ we have

$$f'(z) = u_x(x, y) + v_x(x, y)i = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i = \frac{1}{z},$$

so f can be regarded as a complex analogue of the logarithm function. We will write $\log_S z = f(z)$. If we express each $z \in S$ in the form $z = r_z e^{\theta_z i}$ where $r_z > 0$ and $-\frac{\pi}{2} < \theta_z < \frac{\pi}{2}$, then we have

$$\log_S z = \ln r_z + \theta_z i.$$

We can extend $\log z$ to the set $\mathbb{C} - \{x + 0i \in \mathbb{C} : x < 0\}$ as follows: Define the sets

$$R = \{x + iy \in \mathbb{C} : x < 0, y > 0\} = \{e^{\frac{-\pi}{4} i} z : z \in S\}$$

and

$$T = \{x + iy \in \mathbb{C} : x > 0, y < 0\} = \{e^{\frac{\pi}{4} i} z : z \in S\}.$$

Then the functions $\log_R : R \to \mathbb{C}$ and $\log_T : T \to \mathbb{C}$ defined by

$$\log_R(z) = \log(e^{\frac{-\pi}{4} i} z) + \frac{\pi}{4} i$$

and

$$\log_T(z) = \log(e^{\frac{\pi}{4}i}z) - \frac{\pi}{4}i$$

are holomorphic on R and T by the Chain Rule and are equal to $\log_S z$ on $R \cap S$ and $T \cap S$, respectively. Note also that

$$\log_R'(z) = \frac{1}{e^{\frac{-\pi}{4}i}z}e^{\frac{-\pi}{4}i} = \frac{1}{z}$$

and

$$\log_T'(z) = \frac{1}{e^{\frac{\pi}{4}i}z}e^{\frac{\pi}{4}i} = \frac{1}{z}.$$

We will define $\log : \mathbb{C} - \{x + 0i \in \mathbb{C} : x < 0\} \to \mathbb{C}$ by

$$\log z = \left\{\begin{matrix} \log_R(z) & z \in R \\ \log_S(z) & z \in S \\ \log_T(z) & z \in T \end{matrix}\right\} = \ln r_z + \theta_z i$$

where $z = r_z e^{\theta_z i}$, $r_z > 0$, and $-\pi < \theta_z < \pi$.

Properties of $\log z$:

1. The expression $e^{\log z}$ is defined for all $z \in \mathbb{C} - \{x + 0i \in \mathbb{C} : x < 0\}$. If we write $z = x + iy = re^{i\theta}$ where $-\pi < \theta < \pi$ and $r > 0$, then

$$e^{\log z} = e^{\ln r + i\theta} = e^{\ln r}e^{i\theta} = re^{i\theta} = z.$$

2. The expression $\log(e^z)$ is defined for all $z \in \{x+iy : y \text{ is not an odd multiple of } 2\pi\}$. Given $z = x + iy$ in this set, there is a unique integer n such that

$$z + 2\pi ni = x + iy_0 \in \{x + iy : -\pi < y < \pi\},$$

and

$$\log(e^z) = x + iy_0 = z + 2\pi ni.$$

3. The equation $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ holds provided we can write $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ where $\theta_1, \theta_2, \theta_1 + \theta_2 \in (-\pi, \pi)$, but fails otherwise.

4. For any $\theta \in \mathbb{R}$ we can define a logarithm function

$$\log_\theta : \mathbb{C} - \{re^{i\theta} : r > 0\} \to \mathbb{C}$$

via

$$\log_\theta(z) = \log(e^{(\pi-\theta)i}z).$$

The Chain Rule yields $\log_\theta'(z) = \frac{1}{z}$.

Exponentiation

Definition: Let z and w be complex numbers, and assume $z \notin \{x+0i : x < 0\}$. Then

$$z^w = e^{w \log z}.$$

For example, for $n \in \mathbb{N}$ and $s = \sigma + \tau i$ we have

$$n^s = e^{s \log n} = e^{\sigma \ln n + \tau \ln n i} = n^\sigma(\cos(n^\tau) + \sin(n^\tau)i).$$

Series of Complex Numbers

Definition: Let (a_n) be a sequence of complex numbers. The sequence of partial sums associated with (a_n) is (s_n), where

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

If (s_n) converges to a finite limit s then we say that the series $\sum_{n=0}^\infty a_n$ converges and define

$$\sum_{n=0}^\infty a_n = \lim_{n\to\infty} s_n = s.$$

If the limit does not exist then we say that $\sum_{n=0}^\infty a_n$ diverges.

Example: Let $z \in \mathbb{R}$ be given. Then

$$\sum_{n=0}^\infty z^n = \frac{1}{1-z}$$

if $||z|| < 1$ and diverges if $||z|| \geq 1$. Reason: we have

$$s_n = 1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1-z}.$$

when $z \neq 1$. Divergence if clear if $z = \pm 1$. If $||z|| > 1$ then (s_n) is unbounded, and if $||z|| < 1$ then $s_n \to \frac{1}{1-z}$.

Definition: Let (a_n) be a sequence of complex numbers. We say that $\sum_{n=0}^\infty a_n$ converges absolutely if and only if $\sum_{n=0}^\infty ||a_n||$ converges.

Example: The series $\sum_{n=0}^{\infty} z^n$ converges absolutely for all z satisfying $||z|| < 1$.

Theorem: Absolute convergence implies convergence.

Proof: Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely. Let (s_n) be the sequence of partial sums of (a_n), and let S_n be the sequence of partial sums of $(||a_n||)$. Then we have

$$||s_n - s_m|| = ||a_{m+1} + \cdots + a_n|| \leq ||a_{m+1}|| + \cdots + ||a_n|| = |S_n - S_m|.$$

Since (S_n) converges, it is Cauchy, hence (s_n) is Cauchy, hence (s_n) converges. □

Example: Let $s = \sigma + \tau i \in \mathbb{C}$ where $\sigma > 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely, hence converges.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test but does not converge absolutely by the Integral Comparison Test. Hence convergence does not necessarily imply absolute convergence.

Theorem: Let (a_n) be a sequence of complex numbers and let (s_n) be the associated sequence of partial sums. If (s_n) converges then $a_n \to 0 + 0i$.

Proof: Suppose $s_n \to s$. Then $s_{n-1} \to s$, hence $s_n - s_{n-1} \to 0 + 0i$, hence $a_n \to 0 + 0i$. □

Comparison Test: Let (a_n) be a sequence of complex numbers and let (α_n) be a sequence of positive real numbers. If $\sum_{n=0}^{\infty} \alpha_n$ converges and there exists $\gamma > 0$ and n_0 such that $||a_n|| \leq \gamma \alpha_n$ for all $n \geq n_0$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Proof: Assume that $(||a_n||)$ and (α_n) have partial sums S_n and σ_n, respectively. For $n > m$ we have

$$|S_n - S_m| = ||a_{m+1}|| + \cdots + ||a_n|| \leq \gamma\alpha_{m+1} + \cdots + \gamma\alpha_n = \gamma|\sigma_n - \sigma_m|.$$

Since (σ_n) converges, (σ_n) is Cauchy, therefore (S_n) is Cauchy, therefore (S_n) converges. □

Theorem: Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent. Then

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

as absolutely converent and has limit equal to

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof: We have

$$\left|\left|\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) - \sum_{n=0}^{N}\sum_{k=0}^{n} a_k b_{n-k}\right|\right| \leq$$

$$\left|\left|\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) - \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)\right|\right| +$$

$$\left|\left|\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) - \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right)\right|\right| +$$

$$\left|\left|\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) - \sum_{n=0}^{N}\sum_{k=0}^{n} a_k b_{n-k}\right|\right| \leq$$

$$\left|\left|\sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N} a_n\right|\right| \cdot \sum_{n=0}^{\infty} ||b_n|| +$$

$$\sum_{n=0}^{\infty} ||a_n|| \cdot \left|\left|\sum_{n=0}^{\infty} b_n - \sum_{n=0}^{N} b_n\right|\right| +$$

$$\left(\sum_{r>\frac{N}{2}} ||a_r||\right)\left(\sum_{s=0}^{\infty} ||b_s||\right) + \left(\sum_{r=0}^{\infty} ||a_r||\right)\left(\sum_{s>\frac{N}{2}} ||b_s||\right),$$

which approaches 0 as $N \to \infty$. This establishes the limit. We also have

$$\sum_{n=0}^{\infty}\left|\left|\sum_{k=0}^{n} a_k b_{n-k}\right|\right| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} ||a_k b_{n-k}|| = \left(\sum_{n=0}^{\infty} ||a_n||\right)\left(\sum_{n=0}^{\infty} ||b_b||\right),$$

which proves absolute convergence. $\square$

Rearrangements Theorem : Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then for any permutation $(a_{\pi(n)})$ of (a_n),

$$\sum_{k=1}^{\infty} a_{\pi(n)} = \sum_{k=1}^{\infty} a_k.$$

Proof: For any n such that $\{1, \ldots, N\} \subseteq \{\pi(1), \ldots, \pi(n)\}$ we have

$$\left|\left|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k\right|\right| \leq \sum_{k=N+1}^{\infty} ||a_k||.$$

Choosing N sufficiently large, we can make the difference arbitrarily small, hence

$$\lim_{n\to\infty} \left|\left|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k\right|\right| = 0,$$

hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k = 0,$$

and the result follows. □

Limsup and Liminf

The Root Test and Ratio Test for convergence or divergence of infinite series are defined in terms of the limsup of a sequence. Let (a_n) be a sequence of real numbers. If (a_n) has no upper bound then we say $\limsup_{n\to\infty} a_n = +\infty$. Now assume that (a_n) has a finite upper bound. Then for each n the set

$$A_n = \{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

has a finite least upper bound. Since

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots ,$$

we have

$$\sup A_1 \geq \sup A_2 \geq \sup A_3 \geq \cdots .$$

If $(\sup A_n)$ has no finite lower bound then we way $\limsup_{n\to\infty} a_n = -\infty$. If $(\sup A_n)$ does have a finite lower bound then the sequence converges to a limit. By definition,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup A_n.$$

The liminf of a sequence is defined similarly.

Example: Let $(a_n) = (1, 2 + \frac{1}{2}, 1, 2 + \frac{1}{4}, 1, 2 + \frac{1}{6}, \dots)$. Then

$$(A_n) = (2 + \frac{1}{2}, 2 + \frac{1}{2}, 2 + \frac{1}{4}, 2 + \frac{1}{4}, \dots)$$

and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} A_n = 2.$$

Theorem: When $\lim_{n\to} a_n$ exists, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

Proof: Let A_n be as above. Assume $\lim_{n\to\infty} a_n = a$. Let $\epsilon > 0$ be given. Then there exists N such that

$$a - \frac{\epsilon}{2} < a_N, a_{N+1}, a_{N+2}, \dots < a + \frac{\epsilon}{2},$$

hence for any $n \geq N$ we have

$$a - \frac{\epsilon}{2} < a_n, a_{n+1}, a_{n+2}, \dots < a + \frac{\epsilon}{2},$$

which implies

$$a - \frac{\epsilon}{2} < \sup\{a_n, a_{n+1}, a_{n+2}, \dots\} \leq a + \frac{\epsilon}{2},$$

which implies

$$a - \epsilon < \sup A_n < a + \epsilon.$$

This implies $\lim_{n\to\infty} \sup A_n = a$. $\square$

Root Test: Let (a_n) be a sequence of complex numbers and let

$$\limsup_{n\to\infty} ||a_n||^{\frac{1}{n}} = L.$$

Then:

(1) If $L < 1$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(2) If $L > 1$ then (a_n) is unbounded and $\sum_{n=0}^{\infty} a_n$ diverges.

Proof: Write

$$A_n = \sup\left\{||a_i||^{\frac{1}{i}} : i \geq n\right\}$$

for each n. Then we have $A_1 \geq A_2 \geq \cdots \geq L$ and $A_n \to L$.

(1) Choose any r satisfying $L < r < 1$. Then there exists n such that $A_n < r$. Hence $i \geq n$ implies $||a_i||^{\frac{1}{i}} < r$, which implies $||a_i|| < r^i$. Since $\sum_{n=0}^{\infty} r^n$ converges, $\sum_{n=0}^{\infty} a_n$ converges absolutely by the Comparison Test.

(2) Suppose $L > 1$. Choose r so that $L > r > 1$. For all n we have $A_n > r$, so for each n there exists $n' \geq n$ such that $||a_{n'}||^{\frac{1}{n'}} > r$, which implies $||a_{n'}|| > r^{n'}$. So we can find n_1 such that $||a_{n_1}|| \geq r^{n_1}$, and we can find $n_2 \geq n_1 + 1$ such that $||a_{n_2}|| \geq r^{n_2}$, and we can find $n_3 \geq n_2 + 1$ such that $||a_{n_3}|| \geq r^{n_3}$, etc. Since (r^{n_k}) is unbounded, (a_{n_k}) is unbounded, hence (a_n) is unbounded, hence $a_n \not\to 0$, hence $\sum_{n=0}^{\infty} a_n$ diverges. $\square$

Example: Let (a_n) be any sequence of complex numbers inside the unit circle. Then

$$\left|\left|\frac{a_n}{2^n}\right|\right|^{\frac{1}{n}} \leq \frac{1}{2}$$

for each n, hence

$$\limsup_{n\to\infty} \left|\left|\frac{a_n}{2^n}\right|\right|^{\frac{1}{n}} \leq \frac{1}{2} < 1,$$

hence $\sum_{n=1}^{\infty} \frac{a^n}{2^n}$ converges to a complex number.

Ratio Test: Let (a_n) be a non-zero sequence. Then:

(1) If $\limsup_{n\to\infty} \left|\left|\frac{a_{n+1}}{a_n}\right|\right| < 1$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(2) If $\left|\left|\frac{a_{n+1}}{a_n}\right|\right| \geq 1$ for all $n \geq N$ then $\sum_{n=0}^{\infty} a_n$ diverges.

(3) If $\lim_{n\to\infty} \left|\left|\frac{a_{n+1}}{a_n}\right|\right| > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Proof: (1) Write

$$A_n = \sup\left\{\left|\left|\frac{a_{i+1}}{a_i}\right|\right| : i \geq n\right\}$$

for each n. Then we have $A_1 \geq A_2 \geq \cdots \geq L$ and $A_n \to L$. Choose any r satisfying $L < r < 1$. Then there exists N such that $A_N < r$, which implies $\left|\left|\frac{a_{n+1}}{a_n}\right|\right| < r$ for all $n \geq N$. For any $k \geq 0$ we have

$$||a_{N+k}|| \leq r||a_{N+k-1}|| \leq r^2||a_{N+k-2}|| \leq \cdots \leq r^k||a_N||.$$

In otherwords, for $n \geq N$,

$$||a_n|| \leq r^{n-N}||a_N||.$$

Hence $||a_n|| < cr^n$ for $n \geq N$ where $c = \frac{||a_N||}{r^N}$. Since $\sum_{n=0}^{\infty} r^n$ converges, $\sum_{n=0}^{\infty} a_n$ converges absolutely by the Comparison Test.

(2) The condition implies $||a_n|| \geq ||a_N|| > 0$ for all $n \geq N$, hence $a_n \not\to 0$, hence $\sum_{n=0}^{\infty} a_n$ diverges.

(3) The condition implies that $\left|\left|\frac{a_{n+1}}{a_n}\right|\right| > 1$ beyond a certain point, hence case (2) applies. $\square$

Functions Defined by Power Series

Power series: An expression of the form $\sum_{n=0}^{\infty} a_n z^n$ where $a_n \in \mathbb{C}$ for each n and $z \in \mathbb{C}$. We can define the function $f : S \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ provided the power series converges at each $z \in S$.

A power series always converges at $z = 0$. So power series fall into three categories:

(a) Converges only at $z = 0$.

(b) Converges at some $z_0 \neq 0$ and diverges at some $z_1 \neq 0$.

(c) Converges at every z.

For any particular power series $\sum_{n=0}^{\infty} a_n z^n$, we can determine which case we are in as follows: Let

$$l = \limsup_{n\to\infty} ||a_n||^{\frac{1}{n}}$$

and let

$$l(z) = \limsup_{n\to\infty} ||a_n z^n||^{\frac{1}{n}}.$$

When l is finite,

$$l(z) = l||z||.$$

When $z \neq 0$ and $l = \infty$, $l(z) = \infty$.

(a) Suppose $l = 0$. Then $l(z) = 0$ when we apply the root, hence $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z.

(b) Suppose $0 < l < \infty$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z satisfying $||z|| < \frac{1}{l}$ and $||a_n z^n||$ is unbounded when $z > \frac{1}{l}$. This is case (b).

(c) Suppose $l = \infty$. When $z \neq 0$, $l(z) = \infty$, hence $||a_n z^n||$ is unbounded. This is case (a).

In summary, the power series converges absolutely for all z satisfying $||z|| < \frac{1}{l}$ and diverges for all z satisfying $||z|| > \frac{1}{l}$, interpreting the expression $\frac{1}{l}$ appropriately. We say that $R = \frac{1}{l}$ is the radius of convergence of the power series.

Example: Using the Root Test, the power series $\sum_{n=1}^{\infty} \frac{i^n}{n} z^n$ converges absolutely when $||z|| < 1$ and diverges when $||z|| > 1$. Convergence is conditional on the unit circle: the series converges at $z = i$ by the Alternating Series Test and diverges at $z = -i$ by the Integral Comparison Test.

Functions Defined by Power Series are Infinitely Differentiable

Lemma: $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Proof: Let $\epsilon > 0$ be given. We wish to solve $n^{\frac{1}{n}} < 1 + \epsilon$, or equivalently $n < (1+\epsilon)^n$. It will suffice to solve $n < 1 + n\epsilon + \frac{1}{2}n(n-1)\epsilon^2$. This will be true when $1 < \frac{1}{2}(n-1)\epsilon^2$, i.e. $n > \frac{2}{\epsilon^2}$. $\square$

Theorem: Let (a_n) be a sequence of complex numbers and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n$. Then $f(z)$ and $g(z)$ have the same radius of convergence R and for all z such that $||z|| < R$, $f'(z) = g(z)$.

Proof: Given that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, the Root Test shows that $f(z)$ and $g(z)$ have the same radius of convergence R. Now fix z_0 where $||z_0|| < R$. We will show $f'(z_0) = g(z_0)$.

For each $n \geq 0$ let $s_n(z) = \sum_{k=0}^{n} a_n z^n$. Fix r satisfying $||z_0|| < r < R$. When $||z|| < r$ and $z \neq z_0$ we have

$$\left|\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right|\right| \leq$$

$$\left|\left|\frac{f(z)-f(z_0)}{z-z_0} - \frac{s_n(z)-s_n(z_0)}{z-z_0}\right|\right| + \left|\left|\frac{s_n(z)-s_n(z_0)}{z-z_0} - s_n'(z_0)\right|\right| + ||s_n'(z_0) - g(z_0)||.$$

Given that

$$\frac{f(z)-f(z_0)}{z-z_0} - \frac{s_n(z)-s_n(z_0)}{z-z_0} = \sum_{k=n+1}^{\infty} a_k(z^{k-1} + z^{k-2}z_0 + \cdots + z_0^{k-2}z + z_0^{k-1}),$$

we have

$$\left|\left|\frac{f(z)-f(z_0)}{z-z_0} - \frac{s_n(z)-s_n(z_0)}{z-z_0}\right|\right| \leq \sum_{k=n+1}^{\infty} k||a_k||r^{k-1}.$$

Now let $\epsilon > 0$ be given. Since $g(z)$ converges absolutely at r and $s_n'(z_0)$ converges to $g(z_0)$, there exists n such that

$$\sum_{k=n+1}^{\infty} k||a_k||r^{k-1} < \frac{\epsilon}{3}$$

and

$$||s_n'(z_0) - g(z_0)|| < \frac{\epsilon}{3}.$$

Fixing this value of n, there exists $\delta > 0$ such that $0 < ||z - z_0|| < \delta$ forces both $||z|| < r$ and

$$\left|\left|\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right|\right| < \frac{\epsilon}{3}.$$

Hence $0 < ||z - z_0|| < \delta$ forces

$$\left|\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)\right|\right| < \epsilon.$$

Hence $f'(z_0) = g(z_0)$. □

Complex Line Integrals

Path: A function of the form $\gamma : [a, b] \to \mathbb{C}$ of the form $\gamma(t) = x(t) + y(t)i$.

Definite Integral: Given a path $\gamma : [a, b] \to \mathbb{C}$,

$$\int_a^b \gamma(t)\ dt = \int_a^b x(t)\ dt + \left(\int_a^b y(t)\ dt\right) i.$$

Derivative of a Path: $\gamma'(t) = x'(t) + y'(t)i$, using the one-sided limit to compute $\gamma'(a)$ and $\gamma'(b)$.

Theorem: When γ and Γ are paths on $[a, b]$ and $\Gamma'(t) = \gamma(t)$ on $[a, b]$, then

$$\int_a^b \gamma(t)\ dt = \Gamma(b) - \Gamma(a).$$

Proof: Fundamental theorem of calculus applied to the real and imaginary parts of the integral. □

Line Integral: Given a continuously differentiable path $\gamma : [a, b] \to \mathbb{C}$ and a continuous function $f : S \to \mathbb{C}$ where $\gamma([a, b]) \subseteq S$, the line integral of f over γ is

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

Theorem: Given a continuously differentiable function $\gamma : [a, b] \to \mathbb{C}$ and a holomorphic function $f : S \to \mathbb{C}$ where $\gamma([a, b]) \subseteq S$,

$$\frac{d}{dt}(f(\gamma(t)) = f'(\gamma(t))\gamma'(t)$$

for each $t \in [a, b]$.

Proof: For any $t_0 \in [a, b]$,

$$\Delta_{f \circ \gamma, t_0}(t) = \Delta_{f, \gamma(t_0)}(t) \cdot \Delta_{\gamma, t_0}(t).$$

The formula results from letting $t \to t_0$. □

Corollary: When $F(z)$ is an antiderivative of $f(z)$ along $\gamma([a, b])$,

$$\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: The path $F(\gamma(t))$ is an antiderivative of the path $f(\gamma(t))\gamma'(t)$ along $[a, b]$. □

Corollary: When $f(z)$ has an antiderivative along $\gamma([a, b])$ and $\gamma(a) = \gamma(b)$,

$$\int_\gamma f(z)\ dz = 0.$$

Example: Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(t) = e^{it}$. Then

$$\int_\gamma \frac{1}{z}\ dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\ dt = 2\pi i,$$

hence $\frac{1}{z}$ does not have an antiderivative on $\mathbb{C} - \{0\}$. If we define $\gamma_1 : [0, \pi] \to \mathbb{C}$ by $\gamma_1(t) = e^{it}$ and $\gamma_2 : [\pi, 2\pi]$ by $\gamma_2(t) = e^{it}$, then

$$\int_\gamma \frac{1}{z}\ dz = \int_{\gamma_1} \frac{1}{z}\ dz + \int_{\gamma_2} \frac{1}{z}\ dz =$$

$$\log_{\frac{3\pi}{2}}(z)\Big|_{1}^{-1} + \log_{\frac{\pi}{2}}(z)\Big|_{-1}^{1} =$$

$$\log(-iz)|_{1}^{-1} + \log(iz)|_{-1}^{1} =$$

$$\log(i) - \log(-i) + \log(i) - \log(-i) =$$

$$\frac{\pi}{2}i + \frac{\pi}{2}i + \frac{\pi}{2}i + \frac{\pi}{2}i = 2\pi i.$$

Equivalent Paths

Definition: Two paths $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are equivalent if and only if $\gamma_2 = \gamma_1 \circ s$ where $s : [c, d] \to [a, b]$ is a differentiable bijection satisfying $s'(t) > 0$ for all t.

Theorem: When $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are equivalent,

$$\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz.$$

Proof: Write $f(z) = u(z) + v(z)i$, $\gamma_1(t) = x(t) + iy(t)$. Then

$$\int_{\gamma_2} f(z)\ dz = \int_c^d u(\gamma_1(s(t)))x'(s(t))s'(t) - v(\gamma_1(s(t)))y'(s(t))s'(t)\ dt +$$

$$\left(\int_c^d u(\gamma_1(s(t)))y'(s(t))s'(t) + v(\gamma_1(s(t)))x'(s(t))s'(t)\ dt\right) i.$$

Making the substitution $\theta = s(t)$, $d\theta = s'(t)\ dt$ in the two summands, we obtain

$$\int_a^b u(\gamma_1(\theta))x'(\theta) - v(\gamma_1(\theta))y'(\theta)\ d\theta + \left(\int_a^b u(\gamma_1(\theta))y'(\theta) + v(\gamma_1(\theta))x'(\theta)\ d\theta\right) i =$$

$$\int_{\gamma_1} f(z)\ dz.$$

□

Complex Line Integrals over Piecewise Smooth Paths

For $1 \le i \le n$ let $\gamma_i : [a_i, b_i] \to \mathbb{C}$ be a continuously differentiable path satisfying $\gamma_i(b_i) = \gamma_{i+1}(a_i)$ for $1 \le i \le n-1$. Then we will say that $\gamma = \gamma_1 + \cdots + \gamma_n$ is a piecewise smooth path and define

$$\int_\gamma f(z)\ dz = \sum_{i=1}^n \int_{\gamma_i} f(z)\ dz.$$

Theorem: When $f : S \to \mathbb{C}$ has an antiderivative F defined on the image of a continuous piecewise smooth path γ from z_1 to z_2, then

$$\int_\gamma f(z)\ dz = F(z_2) - F(z_1).$$

Proof: This follows from

$$\int_{\gamma_i} f(z)\ dz = F(\gamma_i(b_i)) - F(\gamma_i(a_i))$$

for each i, where γ_i has domain $[a_i, b_i]$ for each i. $\square$

Corollary: When $f : S \to \mathbb{C}$ has an antiderivative F defined on the image of a closed piecewise-smooth path γ, then

$$\int_\gamma f(z)\ dz = 0.$$

$\square$

<u>**Change of Variables in a Line Integral**</u>

Theorem (Change of Variables): Let $f : S \to \mathbb{C}$ be continuous, and $g : T \to \mathbb{C}$ be holomorphic function, and let $\gamma : [a, b] \to T$ be piecewise smooth. Then

$$\int_{g\circ\gamma} f(z)\ dz = \int_\gamma f(g(z))g'(z)\ dz.$$

Proof: We have

$$\int_{g\circ\gamma} f(z)\ dz = \int_a^b f(g(\gamma(t)))g'(\gamma(t))\gamma'(t)\ dt = \int_\gamma f(g(z))g'(z)\ dz.$$

Example: For any $a \in \mathbb{C}$,

$$\int_\gamma f(a+z)\ dz = \int_\gamma f(g(z))g'(z)\ dz = \int_{a+\gamma} f(z)\ dz$$

using $g(z) = a + z$, where $a + \gamma$ is the translation of γ by a.

Example: For any $a \in \mathbb{C} - \{0\}$, $\int_\gamma f(az)\ dz = \frac{1}{a}\int_\gamma f(g(z))g'(z)\ dz = \frac{1}{a}\int_{a\gamma} f(z)\ dz$ where $a\gamma$ is the dilation of γ by a.

Example: Let $r > 0$ be given, and define $\gamma_r : [0, 2\pi]$ by $\gamma_r(t) = re^{it}$. Then

$$\int_{\gamma_r} \frac{dz}{z-a} = \begin{cases} 0 & r < ||a|| \\ 2\pi i & a > ||a||. \end{cases}$$

Proof: We will start by making the change of variables

$$\int_{\gamma_r} \frac{dz}{z-a} = \int_{a+\gamma_r} \frac{dz}{z}.$$

When $r < ||a||$, $a + \gamma_r$ is a curve entirely contained one of the two vertical half-planes not touching the line $x = 0$, and $\frac{1}{z}$ has an antiderivative on each half-plane. Hence the integral evaluates to zero.

Suppose $r > ||a||$. Then for sufficiently small s the curve γ_s is inside the curve $a + \gamma_r$, and there are two closed piece-wise smooth curves α and β, intersecting along the real axis only and restricted to regions where $\frac{1}{z}$ has an antiderivative, satisfying

$$0 = \int_\alpha \frac{dz}{z} + \int_\beta \frac{dz}{z} = \int_{a+\gamma_r} \frac{dz}{z} - \int_{\gamma_s} \frac{dz}{z}.$$

This implies

$$\int_{a+\gamma_R} \frac{dz}{z} = \int_{\gamma_s} \frac{dz}{z} = 2\pi i.$$

The M–L Inequality

Lemma: Let $\gamma : [a, b] \to \mathbb{C}$ be integrable. Then

$$\left|\left|\int_a^b \gamma(t)\ dt\right|\right| \leq \int_a^b ||\gamma(t)||\ dt.$$

Proof: Write $\int_a^b \gamma(t)\ dt = z$. If $z = 0$ there is nothing to prove. If $z \neq 0$, then

$$||z||^2 = \overline{z}z = \int_a^b \overline{z}\gamma(t)\ dt = \int_a^b \text{re}\ (\overline{z}\gamma(t))\ dt \leq$$

$$\int_a^b ||\overline{z}\gamma(t)||\ dt = ||z|| \int_a^b ||\gamma(t)||\ dt.$$

Now divide by $||z||$. $\square$

Theorem (M–L Inequality): Let $\gamma : [a, b] \to \mathbb{C}$ be continuously differentiable and let f be continuous. Then

$$\left|\left|\int_\gamma f(z)\ dz\right|\right| \le ML$$

where $M = \sup_{z \in \gamma([a,b])} ||f(\gamma(z))||$ and $L = \int_a^b ||\gamma'(t)||\ dt$.

Proof:

$$\left|\left|\int_\gamma f(z)\ dz\right|\right| = \left|\left|\int_a^b f(\gamma(t))\gamma'(t)\ dt\right|\right| \le \int_a^b ||f(\gamma(t))\gamma'(t)||\ \ dt \le M \int_a^b ||\gamma'(t)||\ dt.$$

$\square$

Remarks:

(1) By continuity of f and compactness of $\gamma([a, b])$, $M = ||f(\gamma(w))||$ for some $w \in \gamma([a, b])$.

(2) The expression $L = \int_a^b ||\gamma'(t)||\ dt$ can be interpreted as the length of γ. For example, when $\gamma(t) = z_1 + t(z_2 - z_1)$ on $[0, 1]$ we have

$$L = \int_0^1 ||z_2 - z_1||\ dt = ||z_2 - z_1||,$$

and when $\gamma(t) = z_0 + re^{ti}$ on $[0, 2\pi]$ we have

$$L = \int_0^{2\pi} ||re^{ti}i||\ dt = 2\pi r.$$

(3) Then M–L-inequality generalizes in a natural way to piecewise smooth paths.

Complex Line Integrals over Straight Line Paths

Notation: Given $z, w \in \mathbb{C}$, $\gamma_{z,w} : [0, 1] \to \mathbb{C}$ is the straight path from z to w defined by

$$\gamma_{z,w}(t) = z + t(w - z).$$

Lemma:

(1) When z_3 is a point on the line strictly between z_1 and z_2 then

$$\int_{\gamma_{z_1,z_2}} f(z)\ dz = \int_{\gamma_{z_1,z_3}} f(z)\ dz + \int_{\gamma_{z_3,z_2}} f(z)\ dz.$$

(2)

$$\int_{\gamma_{z_1,z_2}} f(z)\ dz = -\int_{\gamma_{z_2,z_1}} f(z)\ dz.$$

Proof: (1) For any path $\gamma : [0,1] \to \mathbb{C}$ and $0 \le a < b \le 1$ we have

$$\int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_0^1 f(\gamma(a+(b-a)u))\gamma'(a+(b-a)u)(b-a)\ du,$$

where we have made the change of variables $t = a + (b-a)u$. Defining $\widehat{\gamma} : [0,1] \to \mathbb{C}$ by

$$\widehat{\gamma}(u) = \gamma(a+(b-a)u),$$

we have

$$\int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_0^1 f(\widehat{\gamma}(u))\widehat{\gamma}'(u)\ du.$$

This implies

$$\int_a^b f(\gamma_{z_1,z_2}(t))\gamma'_{z_1,z_2}(t)\ dt = \int_{\gamma_{w_1,w_2}} f(z)\ dz$$

where $w_1 = \gamma(a)$ and $w_2 = \gamma(b)$. Now write $z_3 = (1-\lambda)z_1 + \lambda z_2$ where $\lambda \in [0,1]$. Then

$$\int_{\gamma_{z_1,z_3}} f(z)\ dz = \int_0^1 f(\gamma_{z_1,z_2}(t))\gamma'_{z_1,z_2}(t)\ dt =$$

$$\int_0^\lambda f(\gamma_{z_1,z_2}(t))\gamma'_{z_1,z_2}(t)\ dt + \int_\lambda^1 f(\gamma_{z_1,z_2}(t))\gamma'_{z_1,z_2}(t)\ dt =$$

$$\int_{\gamma_{z_1,z_3}} f(z)\ dz + \int_{\gamma_{z_3,z_2}} f(z)\ dz.$$

(2)

$$\int_{\gamma_{z_2 z_1}} f(z)\ dz = \int_0^1 f(z_1 + t(z_2 - z_1))(z_2 - z_1)\ dt =$$

$$-\int_1^0 f(z_1 + (1-u)(z_2 - z_1))(z_2 - z_1)\ du =$$

$$\int_0^1 f(z_2 + u(z_1 - z_2))(z_2 - z_1)\ du = -\int_{\gamma_{z_2, z_1}} f(z)\ dz.$$

□

<u>**Goursat's Theorem**</u>

Definition: A convex subset of $\mathbb{C}$ is any set S with the property that if $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$ then $z + t(z_2 - z_1) \in S$ for $0 \leq t \leq 1$, i.e. that the straight line segment joining z_1 an z_2 is a subset of S.

Goursat's Theorem: Let $S \subseteq \mathbb{C}$ be a convex open set and let z_1, z_2, z_3 the vertices of a triangle contained in S. Let $f : S \to \mathbb{C}$ be holomorphic on S. Then

$$\int_{\gamma_{z_1, z_3}} f(z)\ dz = \int_{\gamma_{z_1, z_2}} f(z)\ dz + \int_{\gamma_{z_2, z_3}} f(z)\ dz.$$

This result lifts the restriction that z_2 be a point on the line between z_1 and z_3, assuming the hypotheses of the theorem are met.

Proof of Goursat's Theorem: Let T denote the triangle. It will suffice to prove

$$\int_T f(z)\ dz = 0.$$

Joining the midpoints of the sides of T we obtain the four triangles $T_{1,1}, T_{1,2}, T_{1,3}, T_{1,4}$. Using the properties of piecewise smooth paths described above, we obtain

$$\int_T f(z)\ dz = \sum_{i=1}^{4} \int_{T_{1,i}} f(z)\ dz.$$

Choose $i_1 \in \{1, 2, 3, 4\}$ such that $\left\| \int_{T_{1,i_1}} f(z)\ dz \right\|$ is maximal. Then

$$\left\| \int_T f(z)\ dz \right\| \leq 4 \left\| \int_{T_{1,i_1}} f(z)\ dz \right\|.$$

Joining the midpoints of the sides of T_{1,i_1} we obtain the four triangles $T_{2,1}, T_{2,2}, T_{2,3}, T_{2,4}$, and

$$\int_{T_{1,i_1}} f(z)\ dz = \sum_{i=1}^{4} \int_{T_{2,i}} f(z)\ dz.$$

Choose $i_2 \in \{1,2,3,4\}$ such that $\left|\left|\int_{T_{2,i_2}} f(z)\ dz\right|\right|$ is maximal. Then

$$\left|\left|\int_{T_{1,i_1}} f(z)\ dz\right|\right| \leq 4 \left|\left|\int_{T_{2,i_2}} f(z)\ dz\right|\right|,$$

hence

$$\left|\left|\int_{T} f(z)\ dz\right|\right| \leq 4^2 \left|\left|\int_{T_{2,i_2}} f(z)\ dz\right|\right|.$$

Keep on going, obtaining a nested sequence of triangles $T_{1,i_1}, T_{2,i_2}, T_{3,i_3}, \ldots$ satisfying

$$\left|\left|\int_{T} f(z)\ dz\right|\right| \leq 4^n \left|\left|\int_{T_{n,i_n}} f(z)\ dz\right|\right|$$

for all n. If we define X_n as the set of all points enclosed by T_{n,i_n}, then each X_n is compact and diam $(X_n) \leq \frac{1}{2^n}p \to 0$ where p is the perimeter of T, hence $\bigcap_{n=1}^{\infty} X_n = \{z_0\}$ for some $z_0 \in T$. Given that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z)$$

where $\psi(z) \to 0$ as $z \to z_0$, and given that the first two terms have an antiderivative, we have

$$\left|\left|\int_{T_{n,i_n}} f(z)\ dz\right|\right| = \left|\left|\int_{T_{n,i_n}} (z - z_0)\psi(z)\ dz\right|\right| \leq M_n L_n$$

where

$$M_n = ||w_n - z_0||||\psi(w_n)|| \leq \frac{1}{2^n}p||\psi(w_n)||$$

for some $w_n \in T_{n,i_n}$ and

$$L_n = \frac{1}{2^n}p.$$

Hence

$$\left|\left|\int_{T} f(z)\ dz\right|\right| \leq ||\psi(w_n)||p^2.$$

We have $w_n \to z_0$ as $n \to \infty$, hence

$$||\psi(w_n)|| \to 0$$

as $n \to \infty$. This implies $\int_T f(z)\ dz = 0$. $\square$

Antiderivative Construction in an Open Convex Set

Morera's Theorem: Let $S \subseteq \mathbb{C}$ be an open and convex set. Let $f : S \to \mathbb{C}$ be continuous on S. If

$$\int_{\gamma_{z_1,z_3}} f(z)\ dz = \int_{\gamma_{z_1,z_2}} f(z)\ dz + \int_{\gamma_{z_2,z_3}} f(z)\ dz$$

for all $z_1, z_2, z_2 \in S$ then then f has the antiderivative F on S, where for a fixed point $z_0 \in S$ we define

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\ dw.$$

Proof: Since S is open, there exists $\epsilon > 0$ such that $h \in B_\epsilon(0)$ implies $B_\epsilon(z) \subseteq S$. By hypothesis, for all $h \in B_\epsilon(0)$ we have

$$F(z+h) - F(z) = \int_{\gamma_{z_0,z+h}} f(w)\ dw - \int_{\gamma_{z_0,z}} f(w)\ dw =$$

$$\int_{\gamma_{z_0,z}} f(w)\ dw + \int_{\gamma_{z,z+h}} f(w)\ dw - \int_{\gamma_{z_0,z}} f(w)\ dw =$$

$$\int_{\gamma_{z,z+h}} f(w)\ dw.$$

We also have

$$hf(z) = \int_{\gamma_{z,z+h}} f(z)\ dw.$$

Therefore we have, for non-zero values of $h \in B_\epsilon(0)$,

$$\frac{F(z+h) - F(z) - hf(z)}{h} = \int_{\gamma_{z,z+h}} \frac{f(w) - f(z)}{h}\ dw,$$

hence

$$\left|\left|\frac{F(z+h) - F(z)}{h} - f(z)\right|\right| \leq ||f(z_h) - f(z)||$$

for some z_h on the line between z and $z+h$ by continuity of f. As $h \to 0$, $z_h \to z$, hence $||f(w_h) - f(z)|| \to 0$. This implies $F'(z) = f(z)$. $\square$

Theorem: Let $S \subseteq \mathbb{C}$ be an open and convex set. Let $f : S \to \mathbb{C}$ be holomorphic on S. Then f has the antiderivative F on S, where for a fixed point $z_0 \in S$ we define

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\ dw.$$

Proof: By Goursat's Theorem, f meets the hypotheses of Morera's Theorem. $\square$

Cauchy's Theorem in an Open Convex Set

Theorem: Let $S \subseteq \mathbb{C}$ be an open and convex set. Let $f : S \to \mathbb{C}$ be holomorphic on S. Then for all closed curves piecewise smooth $\gamma : [a,b] \to S$,

$$\int_\gamma f(z)\ dz = 0.$$

Proof: The function f has an antiderivative on S. $\square$

Remark: The convex hypothesis can be relaxed in specific examples. For example, if S and T are open and convex, $f : S \cup T \to \mathbb{C}$ is holomorphic, and $\gamma = \alpha + \beta$ is a piecewise smooth curve where α is a closed piecewise smooth curve mapping into S and β is a closed piecewise smooth curve mapping into T, then

$$\int_\gamma f(z)\ dz = \int_{\alpha+\beta} f(z)\ dz = 0.$$

Example, page 44: $\int_0^\infty \frac{1-\cos x}{x^2}$ dx $= \frac{\pi}{2}$. Split path down the y-axis, and argue that each closed subpath belongs to open convex set where $\frac{1-e^{iz}}{z^2}$ is holomorphic. To prove that

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon^+} \frac{1-e^{iz}}{z^2}\ dz = \pi,$$

use the following technique: The expression

$$\frac{e^{iz}-1}{z}$$

is a difference quotient of $F(z) = e^{iz}$ and approaches $F'(0) = i$ as $z \to 0$. So we can write

$$\frac{e^{iz} - 1}{z} = i + R(z)$$

where $R(z) \to 0$ as $z \to 0$. This yields

$$\frac{1 - e^{iz}}{z^2} = \frac{-i - R(z)}{z}.$$

Therefore

$$\int_{\gamma_\epsilon^+} \frac{1 - e^{iz}}{z^2}\ dz = -i \int_{\gamma_\epsilon^+} \frac{1}{z}\ dz - \int_{\gamma_\epsilon^+} \frac{R(z)}{z}\ dz = \pi - \int_{\gamma_\epsilon^+} \frac{R(z)}{z}\ dz.$$

Let M_ϵ be the maximum value of $||R(z)||$ on γ_ϵ^+. By the M-L inequality,

$$\left|\left|\int_{\gamma_\epsilon^+} \frac{R(z)}{z}\ dz\right|\right| \leq M_\epsilon \cdot \frac{1}{\epsilon} \cdot \pi\epsilon = \pi M_\epsilon \to 0$$

as $\epsilon \to 0$. This yields the desired result.

Cauchy's Integral Formula

Notation: Fix $r > 0$ and $a \in \mathbb{C}$. Then

$$C_r(a) = \{z \in \mathbb{C} : ||z - a|| = r\},$$

$$D_r(a) = \{z \in \mathbb{C} : ||z - a|| \leq r\},$$

and

$$\int_{C_r(a)} f(z)\ dz$$

denotes the line integral over the path $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) = a + re^{ti}.$$

Theorem (Cauchy's Integral Formula): Let $S \subseteq \mathbb{C}$ be an open and convex set containing $D_r(a)$. Let $f : S \to \mathbb{C}$ be holomorphic on $D_r(a)$. Then for all $z \in B_r(a)$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(w)}{w - z}\ dw.$$

Proof: Fix $z \in B_r(a)$ and $s \in \mathbb{R}$, $0 < s < r$. The expression $\frac{f(w)}{w-z}$ is a holomorphic function of w on $B_r(a) - \{a\}$ by the quotient rule, and it is possible to define four piecewise smooth close curves $\gamma_1, \gamma_2, \gamma_3, \gamma_r$ restricted to convex open subsets of S that satisfy

$$0 = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \frac{f(w)}{w-z}\ dw = \int_{C_r(a)} \frac{f(w)}{w-z}\ dw - \int_{C_s(z)} \frac{f(w)}{w-z}\ dw$$

Therefore

$$\int_{C_r(a)} \frac{f(w)}{w-z}\ dw = \lim_{s\to 0} \int_{C_s(z)} \frac{f(w)}{w-z}\ dw.$$

On $C_s(z)$ we have

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z} + \Delta_{f,z}(w),$$

hence

$$\int_{C_s(z)} \frac{f(w)}{w-z}\ dw = \int_{C_s(z)} \frac{f(w)}{w-z}\ dw + \int_{C_s(z)} \Delta_{f,z}(w)\ dw = 2\pi f(z) i + \int_{C_s(z)} \Delta_{f,z}(w)\ dw,$$

hence

$$\int_{C_r(a)} \frac{f(w)}{w-z}\ dw = 2\pi f(z) i + \lim_{s\to 0} \int_{C_s(z)} \Delta_{f,z}(w)\ dw.$$

By the M–L-inequality,

$$\left|\left|\int_{C_s(z)} \Delta_{f,z}(w)\ dw\right|\right| \leq 2\pi s ||\Delta_{f,z}(w_s)||$$

for some $w_s \in C_s(z)$. As $s \to 0$, $w_s \to z$, hence $\Delta_{f,z}(w_s) \to f'(z)$, hence $2\pi s||\Delta_{f,z}(w_s)|| \to 0$, hence

$$\left|\left|\int_{C_s(z)} \Delta_{f,z}(w)\ dw\right|\right| \to 0.$$

This implies

$$\int_{C_r(a)} \frac{f(w)}{w-z}\ d{=}2\pi f(z) i.$$

$\square$

Remark: Let $S \subseteq \mathbb{C}$ be an open and convex set containing $D_r(a)$. Let $f : S \to \mathbb{C}$ be holomorphic on $D_r(a)$. Then for all $z \in B_r(a)$ and for all w satisfying $||w - a|| = r$,

$$\frac{w-a}{w-z} = \frac{1}{1 - \frac{z-a}{w-a}} = \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n,$$

hence by Cauchy's Formula

$$f(z) = \frac{1}{2\pi i} \int_{C_r(a)} \left(\sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}} (z-a)^n \right) \, dw.$$

We would like to exchange the order of integration and summation, but we must do this carefully. Hence we make a digression into sequences of functions.

Sequences of Functions

Definition: The norm of a function $f : S \to \mathbb{C}$ is

$$||f|| = \sup\{||f(z)|| : z \in S\}.$$

Definition: Let S be a subset of $\mathbb{C}$ and for each $n \geq 0$ let $f_n : S \to \mathbb{C}$ be a function. We say that (f_n) converges uniformly if and only there exists a function $f : S \to \mathbb{C}$ such that

$$\lim_{n \to \infty} ||f_n - f|| = 0,$$

in which case we say that (f_n) converges uniformly to f.

Theorem: If (f_n) converges uniformly to f on S and each f_n is continuous on S, then f is continuous on S.

Proof: Fix $z_0 \in S$ and let $\epsilon > 0$ be given. For any $z \in S$ we have

$$||f(z) - f(z_0)|| \leq$$

$$||f(z) - f_n(z)|| + ||f_n(z) - f_n(z_0)|| + ||f_n(z_0) - f(z_0)|| \leq$$

$$2||f_n - f|| + ||f_n(z) - f_n(z_0)||$$

for all n. We can choose N so that $||f_N - f|| < \frac{\epsilon}{4}$. Having fixed N, we have

$$||f(z) - f(z_0)|| < \frac{\epsilon}{2} + ||f_N(z) - f(z_0)||.$$

By continuity of f_N, there exists $\delta > 0$ such that for all $z \in S$ satisfying $||z - z_0|| < \delta$, $||f_N(z) - f_N(z_0)|| < \frac{\epsilon}{2}$. Hence

$$||z - z_0|| < \delta \implies ||f(z) - f(z_0)|| < \epsilon$$

for all $z \in S$. □

Weierstrass M-Test: Let S be a subset of $\mathbb{C}$, and for each $n \geq 0$ let $f_n : S \to \mathbb{C}$ be a function. If

$$\sum_{n=0}^{\infty} ||f_n|| = M < \infty$$

then $\sum_{n=0}^{\infty} f_n(z)$ converges to a complex number for each $z \in S$ and the sequence of functions $(\sum_{k=0}^{n} f_k)$ converges uniformly to the function $f : S \to \mathbb{C}$ defined by

$$f(z) = \sum_{n=0}^{\infty} f_n(z).$$

Proof: For any given $z \in S$, $||f_n(z)|| \leq ||f_n||$, and since $\sum_{n=0}^{\infty} ||f_n||$ converges, $\sum_{n=0}^{\infty} f_n(z)$ converges by the Comparison Test. For any $n > m$ and $z \in S$ we have

$$\left|\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{m} f_k(z)\right|\right| = ||f_{m+1}(z) + \cdots + f_n(z)|| \leq$$

$$||f_{m+1}(z)|| + \cdots + ||f_n(z)|| \leq ||f_{m+1}|| + \cdots + ||f_n|| \leq M - \sum_{k=0}^{m-1} ||f_k||.$$

Hence the sequence of partial sums is Cauchy and converges. Fixing m and letting $n \to \infty$,

$$\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{m} f_k(z) \to f(z) - \sum_{k=0}^{m} f_k(z),$$

hence

$$\left|\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{m} f_k(z)\right|\right| \to \left|\left|f(z) - \sum_{k=0}^{m} f_k(z)\right|\right|,$$

hence

$$\left|\left|f(z) - \sum_{k=0}^{m} f_k(z)\right|\right| \leq M - \sum_{k=0}^{m-1} ||f_k||.$$

Since this holds for all $z \in S$,

$$\left|\left|f - \sum_{k=0}^{m} f_k\right|\right| \leq M - \sum_{k=0}^{m-1} ||f_k||.$$

Since

$$M - \sum_{k=0}^{m-1} ||f_k|| \to 0$$

as $m \to \infty$,

$$\left|\left|f - \sum_{k=0}^{m} f_k\right|\right| \to 0$$

as $m \to \infty$. This implies $(\sum_{k=0}^{n} f_k)$ converges to f uniormly on S. $\square$

Theorem: Let S be a subset of $\mathbb{C}$, let $\gamma : [a, b] \to S$ be piecewise smooth, and for each n let $f_n : S \to \mathbb{C}$ be continuous. If (f_n) converges uniformly to f on S then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\ dz = \int_\gamma f(z)\ dz.$$

Proof: It suffices to prove that

$$\left|\left|\int_\gamma f(z)\ dz - \int_\gamma f_n(z)\ dz\right|\right| \to 0.$$

We have

$$\left|\left|\int_\gamma f(z)\ dz - \int_\gamma f_n(z)\ dz\right|\right| = \left|\left|\int_\gamma f(z) - f_n(z)\ dz\right|\right| \leq$$

$$\int_\gamma ||f_n(z) - f(z)||\ \ dz \leq \int_\gamma ||f_n - f||\ dz = ||f_n - f||L \to 0$$

where L is the length of γ. $\square$

Corollary: Let S be a subset of $\mathbb{C}$, let $\gamma : [a, b] \to S$ be piecewise smooth, and for each $n \geq 0$ let $f_n : S \to \mathbb{C}$ be continuous. If

$$\sum_{n=0}^{\infty} ||f_n|| = M < \infty$$

then

$$\sum_{n \to \infty} \left(\int_\gamma f_n(z)\ dz \right) = \int_\gamma \left(\sum_{n=0}^{\infty} f_n(z) \right)\ dz.$$

Proof: Since $(\sum_{k=0}^{n} f_k)$ converges uniformly to the function f defined by

$$f(z) = \sum_{n=0}^{\infty} f_n(z),$$

we have by the previous theorem

$$\sum_{n=0}^{\infty} \left(\int_\gamma f_n(z)\ dz \right) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \int_\gamma f_n(z)\ dz \right) = \lim_{n\to\infty} \int_\gamma \left(\sum_{k=0}^{n} f_n(z) \right)\ dz =$$

$$\int_\gamma f(z)\ dz = \int_\gamma \left(\sum_{n=0}^{\infty} f_n(z)\ dz \right).$$

$\square$

Power Series Expansion of Holomorphic Functions

Theorem: Let $S \subseteq \mathbb{C}$ be an open and convex set containing $D_r(a)$. Let $f : S \to \mathbb{C}$ be holomorphic on $D_r(a)$. Then for all $z \in B_r(a)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(w)}{(w-a)^{n+1}}\ dw.$$

Proof: Fix $z \in B_r(a)$. As we argued above, by Cauchy's Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{C_r(a)} \left(\sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}} (z-a)^n \right) \, dw.$$

For each $n \geq 0$ let $f_n : C_r(a) \to \mathbb{C}$ be defined by

$$f_n(w) = \frac{f(w)}{(w-a)^{n+1}} (z-a)^n.$$

Then

$$||f_n|| = \sup \left\{ \left|\left| \frac{f(w)}{(w-a)^{n+1}} (z-a)^n \right|\right| : w \in C_r(a) \right\} = ||f|| \left(\frac{||z-a||}{r} \right)^{n+1},$$

hence $\sum_{n=0}^{\infty} ||f_n||$ converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{||z-a||}{r} \right)^n.$$

By the last result proved in the section on sequences of functions, this implies

$$f(z) = \sum_{n=0}^{\infty} \left(\int_{C_r(a)} \frac{f(w)}{(w-a)^{n+1}} (z-a)^n \, dw \right) =$$

$$\sum_{n=0}^{\infty} \left(\int_{C_r(a)} \frac{f(w)}{(w-a)^{n+1}} \, dw \right) (z-a)^n.$$

□

Corollary: Let $S \subseteq \mathbb{C}$ be an open and convex set containing $D_r(a)$. Let $f : S \to \mathbb{C}$ be holomorphic on $D_r(a)$. Then f is infinitely differentiable on $B_r(a)$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_r(a)} \frac{f(w)}{(w-a)^{n+1}} \, dw$$

for all $n \geq 0$, and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

for all $z \in B_r(a)$.

Proof: These statements follow from the fact that $f(z)$ has a power series expansion in $B_r(a)$. We showed earlier that functions define by power series are infinitely differentiable. $\square$

Remark: Assume f is holomorphic on $B_r(a)$. For any s satisfying $0 < s < r$, f is holomorphic on $D_s(a)$, hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

for all $z \in B_s(a)$. Since $B_r(a) = \bigcup_{0<s<r} B_s(a)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

for all $z \in B_r(a)$. Hence f is infinitely differentiable on $B_r(a)$. Choosing any s satisfying $0 < s < r$, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_s(a)} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

for all $n \geq 0$.

Remark: It is now possible to derive the power series expansions

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!},$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$. We also have

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$$

for all $z \in B_1(1+0i)$ and

$$\frac{1}{(z-c)^k} = \frac{1}{(a-c)^k} \sum_{n=0}^{\infty} \frac{1}{(c-a)^n} \binom{n+k-1}{k-1} (z-a)^n$$

for all $z \in B_{||c-a||}(a)$.

Power Series Expansions of Products and Quotients

Theorem: Let f and g be holomorphic on $B_r(a)$ and have power series expansions

$$f(z) = \sum_{n=0}^{\infty} f_n (z-a)^n$$

and

$$g(z) = \sum_{n=0}^{\infty} g_n (z-a)^n$$

respectively. Then fg is holomorphic on $B_r(a)$ and has power series expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) (z-a)^n.$$

Proof: We have

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(a)}{n!} (z-a)^n.$$

The product rule and induction yield

$$(fg)^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z),$$

hence

$$\frac{(fg)^{(n)}(a)}{n!} = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \frac{g^{(n-k)}(a)}{(n-k)!} = \sum_{k=0}^{n} f_k g_{n-k}.$$

Matrix Computation of Power Series Products and Quotients

Definition: Let $a(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent in $B_r(0)$. Then we define

$$M_n(a) = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}.$$

Theorem: Let $a(z) = \sum_{n=0}^{\infty} a_n z^n$ and $b(z) = \sum_{n=0}^{\infty} b_n z^n$ be convergent in $B_r(0)$. Then $c(z) = a(z)b(z)$ is convergent in $B_r(0)$ and

$$M_n(c) = M_n(a)M_n(b).$$

Proof: The functions $a(z)$ and $b(z)$ are holomorphic on $B_r(0)$. The function $c(z) = a(z)b(z) = \sum_{n=0}^{\infty} c_n z^n$ is holomorphic on $B_r(0)$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Now let $n \geq \mathbb{N}$ be given. For $0 \leq i, j \leq n$ define

$$\alpha_{i,j} = \begin{cases} a_{i-j} & i \geq j \\ 0 & i < j, \end{cases}$$

$$\beta_{i,j} = \begin{cases} b_{i-j} & i \geq j \\ 0 & i < j, \end{cases}$$

$$\gamma_{i,j} = \begin{cases} c_{i-j} & i \geq j \\ 0 & i < j. \end{cases}$$

Fixing i and j,

$$\sum_{k=0}^{n} \alpha_{i,k}\beta_{k,j} = \sum_{j \leq k \leq i} a_{i-k} b_{k-j} = \sum_{0 \leq p \leq i-j} a_{i-j-p} b_p = \left\{ \begin{matrix} c_{i-j} & i \geq j \\ 0 & i < j \end{matrix} \right\} = \gamma_{i,j}.$$

This implies

$$(\alpha_{i,j})(\beta_{i,j}) = (\gamma_{i,j}),$$

which implies

$$M_n(a)M_n(b) = M_n(c).$$

$\square$

Corollary: Let $a(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic and non-zero in $B_r(a)$. Then

$$M_n(\frac{1}{a}) = M_n(a)^{-1}.$$

Proof: This follows from $a(z)\frac{1}{a(z)} = 1$ and

$$M_n(1) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

$\square$

Example: Let $c(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(2n)!}$ and $s(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(2n+1)!}$. Both functions are holomorphic at all $z \in \mathbb{C}$, and $\cos(z) = c(z^2)$ and $\sin(z) = zs(z^2)$. Since $\sin(z) = 0$ if and only if z is an odd multiple of π, $s(z)$ is non-zero on $B_{\sqrt{\pi}}(0)$. We have

$$M_3(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{24} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{720} & \frac{1}{24} & -\frac{1}{2} & 1 \end{bmatrix}, \qquad M_3(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{6} & 1 & 0 & 0 \\ \frac{1}{120} & -\frac{1}{6} & 1 & 0 \\ -\frac{1}{5040} & \frac{1}{120} & -\frac{1}{6} & 1 \end{bmatrix},$$

$$M_3(s)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{6} & 1 & 0 & 0 \\ \frac{7}{360} & \frac{1}{6} & 1 & 0 \\ \frac{31}{15120} & \frac{7}{360} & \frac{1}{6} & 1 \end{bmatrix}, \qquad M_3(c)M_3(s)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{45} & -\frac{1}{3} & 1 & 0 \\ -\frac{2}{945} & -\frac{1}{45} & -\frac{1}{3} & 1 \end{bmatrix}.$$

This implies that

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots$$

and

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 - \cdots$$

on $B_\pi(0) - \{0\}$.

Liouville's Theorem and The Fundamental Theorem of Algebra

Definition: A function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic at every $z \in \mathbb{C}$ is called entire.

Theorem: A bounded and entire holomorphic function is constant.

Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on $\mathbb{C}$ and satisfies $||f|| = M < \infty$. Let $a \in \mathbb{C}$ be given. By Cauchy's Formula and the M–L-inequality, for all $r > 0$ we have

$$||f'(a)|| = \left|\left| \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(w)}{(w-a)^2}\ dw \right|\right| \leq \frac{1}{2\pi} \frac{M}{r^2} 2\pi r = \frac{M}{r}.$$

Hence $f'(a) = 0$. Since $f'(z)$ is identically zero on $\mathbb{C}$, it is a constant function by Exercise 26, page 31. $\square$

Corollary: Every polynomial $p(z)$ of degree ≥ 1 with complex coefficients has a root in $\mathbb{C}$.

Proof: Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{z^n}{p(z)}$$

is entire. It is bounded: write $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then

$$z^n p(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$

hence

$$\lim_{z \to 0} z^n p(1/z) = a_n,$$

hence there exists $\delta > 0$ such that

$$0 < ||z|| < \delta \implies ||z^n p(1/z) - a_n|| < \frac{1}{2}||a_n||,$$

hence

$$0 < ||z|| < \delta \implies \frac{1}{||f(1/z)||} = ||z^n p(1/z)|| = ||a_n - (z^n p(1/z) - a_n)||$$

$$\geq ||a_n|| - ||z^n p(1/z) - a_n|| > \frac{1}{2}||a_n||,$$

$$||z|| > \frac{1}{\delta} \implies ||f(z)|| < \frac{2}{||a_n||}.$$

Since f is continuous and $D_{\frac{1}{\delta}}(0)$ is compact, $f(z)$ attains a maximum value of $f(z_0)$ on $D_{\frac{1}{\delta}}(0)$. Hence

$$||f(z)|| \leq \max\left(\frac{2}{||a_n||}, f(z_0)\right)$$

for all $z \in \mathbb{C}$. By Liouville's theorem, this implies that $f(z)$ is constant. Hence $p(z) = cz^n$ for some $c \in \mathbb{C}$, contradicting the fact that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Therefore $p(z) \neq 0$ for some $z \in \mathbb{C}$. $\square$

Corollary: Every non-constant polynomial $p(z)$ of degree ≥ 1 with complex coefficients factors into linear factors.

Proof: We can prove this by induction on the degree of $p(z)$, using the fact that if $p(c) = 0$ then $p(z) = q(z)(z - c)$ for some polynomial $q(z)$ of lower degree. $\square$

Laurent Series

Definition: A Laurent Series is an expression of the form $\sum_{n\in\mathbb{Z}} c_n(z-a)^n$ where $a \in \mathbb{C}$ and $c_n \in \mathbb{C}$ for each $n \in \mathbb{Z}$. We say that the Laurent series converges at z if and only if the two infinite series $\sum_{n=0}^{\infty} c_n(z-a)^n$ and $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converge, in which case we define

$$\sum_{n\in\mathbb{Z}} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}.$$

Example: A Laurent series expansion for $(z-a)e^{\frac{1}{z-a}}$ in powers of $z - a$ on $\mathbb{C} - \{a\}$ is given by

$$(z-a)e^{\frac{1}{z-a}} = 1(z-a) + 1(z-a)^0 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}(z-a)^{-n}.$$

Example: Consider the function $f : \mathbb{C} - \{0, i, -i\}$ defined by

$$f(z) = \frac{z+1}{z^4+z^2}.$$

A partial fraction decomposition yields

$$f(z) = \frac{p}{z^2} + \frac{q}{z} + \frac{r}{z-i} + \frac{s}{z+i}$$

where $p = 1$, $q = 1$, $r = \frac{1}{2}(-1+i)$, and $s = \frac{1}{2}(-1-i)$. A Laurent series expansion for $f(z)$ in powers of z is

$$f(z) = pz^{-2} + qz^{-1} + \sum_{n=0}^{\infty} (ra_n + sb_n)z^n$$

for all $z \in B_1(0) - \{0\}$, where $\frac{1}{z-i} = \sum_{n=0}^{\infty} a_n z^n$ and $\frac{1}{z+i} = \sum_{n=0}^{\infty} b_n z^n$. On the other hand, a Laurent series expansion of $f(z)$ in powers of $z - i$ is

$$f(z) = r(z-i)^{-1} + \sum_{n=0}^{\infty} (pa_n + qb_n + sc_n)(z-i)^n$$

for all $z \in B_1(i) - \{i\}$, where $\frac{1}{z^2} = \sum_{n=0}^{\infty} a_n (z-i)^n$, $\frac{1}{z} = \sum_{n=0}^{\infty} b_n (z-i)^n$, and $\frac{1}{z+i} = \sum_{n=0}^{\infty} c_n (z-i)^n$.

The Residue Theorem

Definition: Let $f : B_r(a) - \{a\} \to \mathbb{C}$ have a Laurent series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n.$$

The residue of f at a with respect to this expansion is

$$\mathrm{res}_a f = c_{-1}.$$

Theorem: Assume that $f : B_r(a) - \{a\} \to \mathbb{C}$ is holomorphic and has Laurent series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n.$$

Then for any $0 < s < r$,

$$\int_{C_s(a)} f(z)\, dz = 2\pi i \ \mathrm{res}_a f.$$

Proof: We have $f(z) = g(z) + h(z)$ where

$$g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

and

$$h(z) = \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}.$$

Since $g(z)$ converges on $B_r(a)$, it is holomorphic on $B_r(a)$, hence $h(z) = f(z) - g(z)$ is holomorphic on $B_r(a) - \{a\}$. Hence both g and h are continuous on $B_r(a) - \{a\}$, and we have

$$\int_{C_s(a)} f(z)\ dz = \int_{C_s(a)} g(z)\ dz + \int_{C_s(a)} h(z)\ dz.$$

By Cauchy's Theorem in an open convex set,

$$\int_{C_s(a)} g(z)\ dz = 0.$$

Therefore

$$\int_{C_s(a)} f(z)\ dz = \int_{C_s(a)} h(z)\ dz = \int_{C_s(0)} h(z+a)\ dz =$$

$$\int_{C_s(0)} \left(\sum_{n=1}^{\infty} c_{-n} z^{-n} \right)\ dz.$$

We wish to exchange the order of integration and summation.

Choose any t satisfying $0 < t < s$. Since $\sum_{k=1}^{\infty} c_{-k} t^{-k}$ converges, $\sum_{k=0}^{\infty} c_{-k} z^{k}$ converges absolutely on $C_{\frac{1}{t}}(0)$, hence $\sum_{k=1}^{\infty} c_{-k} z^{-k}$ converges absolutely on $\{z \in \mathbb{C} : ||z|| > t\}$. By the Weierstrass M-test, $\sum_{k=1}^{\infty} c_{-k} z^{-k}$ is the uniform limit of the sequence of functions $(\sum_{k=1}^{p} c_{-k} z^{-k})$ on $\{z \in \mathbb{C} : ||z|| > t\}$, and since $C_s(0) \subseteq \{z \in \mathbb{C} : ||z|| > t\}$, we have

$$\int_{C_s(0)} \left[\sum_{n=1}^{\infty} c_{-n} z^{-n} \right]\ dz = \sum_{n=1}^{\infty} \left[\int_{C_s(0)} c_{-n} z^{-n}\ dz \right] = \int_{C_s(0)} c_{-1} z^{-1}\ dz =$$

$$2\pi i c_{-1} = 2\pi i\ \mathrm{res}_a f.$$

$\square$

Computing Residues

I. If $f(z)$ is holomorphic in $B_r(z_0) - \{z_0\}$ and $f(z) = \frac{1}{(z-a)^n} g(z)$ where $g(z)$ is holomorphic in $B_r(z_0)$, then $g(z)$ has a power series expansion

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

which yields the Laurent Series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k-n}.$$

This yields

$$\mathrm{res}_{z_0} f = a_{n-1} = \frac{1}{(n-1)!} g^{(n-1)}(z_0).$$

Example: Let $f : \mathbb{C} - \{0, i, -i\}$ be defined by

$$f(z) = \frac{z+1}{z^4 + z^2}.$$

We will compute the residue of f at $z_0 = 0, i, -i$. Observe that we have

$$f(z) = \frac{z+1}{z^2(z+i)(z-i)}.$$

Residue at $z_0 = 0$: We have

$$f(z) = \frac{1}{z^2}\left(\frac{z+1}{z^2+1}\right),$$

$$\mathrm{res}_0 f = \frac{1}{1!}\left(\frac{z+1}{z^2+1}\right)'(0) = \left(\frac{-z^2 - 2z + 1}{(z^2+1)^2}\right)(0) = 1.$$

Residue at $z_0 = i$: We have

$$f(z) = \frac{1}{z-i}\left(\frac{z+1}{z^2(z+i)}\right),$$

$$\mathrm{res}_i f = \frac{1}{0!}\left(\frac{i+1}{i^2(i+i)}\right) = -\frac{1}{2} + \frac{i}{2}.$$

Residue at $z_0 = -i$: We have

$$f(z) = \frac{1}{z+i}\left(\frac{z+1}{z^2(z-i)}\right),$$

$$\mathrm{res}_{-i} f = \frac{1}{0!}\left(\frac{-i+1}{(-i)^2(-i-i)}\right) = -\frac{1}{2} - \frac{i}{2}.$$

II. We can compute residues by working with Laurent series directly. For example, consider $f(z) = \frac{e^{az}}{1+e^z}$ where $a \in \mathbb{R}$. We will compute the residue at $z = \pi i$. Expanding the denominator in powers of $z - \pi$ we obtain

$$1 + e^z = 1 + e^{\pi i} e^{z-\pi i} = 1 - \sum_{n=0}^{\infty} \frac{(z-\pi i)^n}{n!} = -(z-\pi i)\sum_{n=1}^{\infty} \frac{(z-\pi i)^{n-1}}{n!}.$$

This yields, for $z \neq \pi i$,

$$f(z) = \frac{1}{z-\pi i}\frac{-e^{az}}{g(z)}$$

where

$$g(z) = \sum_{n=1}^{\infty} \frac{(z-\pi i)^{n-1}}{n!}.$$

Since $g(\pi i) = 1$, $g(z) \neq 0$ on some sufficiently small neighborhood $B_\epsilon(\pi i)$, hence $\frac{-e^{az}}{g(z)}$ is holomorphic on $B_\epsilon(\pi i)$. This implies

$$\mathrm{res}_{\pi i} f = \frac{1}{0!}\frac{-e^{a\pi i}}{g(\pi i)} = -e^{a\pi i}.$$

Generalized Residue Theorem

Theorem: Let $f : S \to \mathbb{C}$ be holomorphic on $S - \{a_1, \ldots, a_n\}$. Assume that f has a Laurent series expansions in powers of $z - a_k$ in $B_{r_k}(a_k) - \{a_k\}$ for each i, that $0 < s_k < r_k$ for each k, and that there exist closed piecewise smooth paths $\gamma_0, \ldots, \gamma_N$ restricted to open and convex subsets of $S - \{a_1, \ldots, a_n\}$ such that

$$\int_{\gamma_0+\cdots+\gamma_N} f(z)\ dz = \int_{\gamma_0} f(z)\ dz - \int_{C_{s_1}(a_1)+\cdots+C_{s_n}(a_n)} f(z)\ dz.$$

Then

$$\int_{\gamma_0} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{a_k} f.$$

Proof: Since f has an antiderivative on γ_k for each k,

$$\int_{\gamma_0+\cdots+\gamma_N} f(z)\ dz = 0.$$

Hence

$$\int_{\gamma_0} f(z)\ dz = \int_{C_{s_1}(a_1)+\cdots+C_{s_n}(a_n)} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{a_k} f.$$

Example: Let $f : \mathbb{C} - \{0, i, -i\} \to \mathbb{C}$ be defined by $f(z) = \frac{z+1}{z^4+z^2}$. Then

$$\int_{C_{\frac{3}{2}}(i)} \frac{z+1}{z^4+z^2}\ dz = 2\pi\left(\mathrm{res}_0 f + \mathrm{res}_i f\right) = -\pi + \pi i.$$

Trigonometric Integrals

Let $c(z) = \frac{z+1/z}{2} = \frac{z^2+1}{2z}$, let $s(z) = \frac{z-1/z}{2i} = \frac{z^2-1}{2iz}$, and let $f(z, y)$ be a real-valued function. Then

$$\int_{C_1(0)} f(c(z), s(z)) z^{n-1}\ dz = i\int_0^{2\pi} f(\cos\theta, \sin\theta)(\cos n\theta + i\sin n\theta)\ d\theta.$$

Comparing real and imaginary parts,

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\cos n\theta\ d\theta = \mathrm{im} \int_{e^{i\theta}} f(c(z), s(z)) z^{n-1}\ dz$$

and

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\sin n\theta\ d\theta = -\mathrm{re} \int_{e^{i\theta}} f(c(z), s(z)) z^{n-1}\ dz.$$

So for example

$$\int_0^{2\pi} \cos^4\theta\ d\theta = \mathrm{im} \int_{C_1(0)} \left(\frac{z^2+1}{2z}\right)^4 \frac{1}{z}\ dz =$$

$$\mathrm{im}\ 2\pi i \cdot \mathrm{res}_0 \left(\frac{z^2+1}{2z}\right)^4 \frac{1}{z} = \mathrm{im}\ 2\pi i \cdot \mathrm{res}_0 \frac{z^8+4z^6+6z^4+4z^2+1}{16z^5} = \frac{3\pi}{4}.$$

Improper Integrals

Let $f(x)$ be a complex-valued function on $(-\infty, \infty)$. Then by definition

$$\int_0^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_a^R f(x)\ dx,$$

$$\int_{-\infty}^{0} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{a} f(x)\ dx,$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{\infty} f(x)\ dx + \int_{-\infty}^{0} f(x)\ dx,$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx,$$

assuming the limits exist. When $\int_{-\infty}^{\infty} f(x)\ dx$ exists,

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx.$$

Example: The improper integral $\int_0^\infty \frac{1}{1+x^3}\ dx$ exists: write $\int_0^1 \frac{1}{1+x^3}\ dx = a$. Then for each $R \geq 1$,

$$\int_0^R \frac{1}{1+x^3}\ dx = a + \int_1^R \frac{1}{1+x^3}\ dx < a + \int_1^R \frac{1}{x^3}\ dx =$$

$$a + \frac{1}{2} - \frac{1}{2R^2} \leq a + \frac{1}{2}.$$

Hence the sequence $(\int_0^n \frac{1}{1+x^3}\ dx)$ is increasing and bounded above by $a + \frac{1}{2}$, hence converges to a finite limit L. Therefore

$$\int_0^\infty \frac{1}{1+x^3}\ dx = L.$$

Improper Integrals and Semicircular Paths

I. Suppose that $f(z)$ is holomorphic on the real axis and at all but a finite number of points $\{a_1, \ldots, a_n\}$ above the real axis. Then integrating $f(z)$ around the piecewise smooth path $\alpha_R + \beta_R$ where $\alpha_R(x) = x$ on $[-R, R]$ and $\beta_R(t) = Re^{it}$ on $[0, \pi]$ we obtain

$$\int_{-R}^{R} f(x)\ dx + \int_{\beta_R} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{res}_{a_k} f.$$

See the figure on page 79. Let $||f||_R$ denote the maximum value of $||f(z)||$ on $C_R(0)$. Then

$$\lim_{R\to\infty} R||f||_R = 0 \implies$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \text{res}_{a_k} f.$$

Example: The function $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ is holomorphic on the real axis and at all points except $z = i$ above the real axis. Moreover when $||z|| = R > 1$ we have

$$\left|\left|\frac{1}{1+z^2}\right|\right| \le \frac{1}{R^2-1},$$

hence $R||f||_R \le \frac{R}{R^2-1} \to 0$ as $R \to \infty$. Given that

$$\text{res}_i f = \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i},$$

we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{2\pi i}{2i} = \pi.$$

Since $\frac{1}{1+x^2}$ is an even function, this implies

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{2\pi i}{2i} = \frac{\pi}{2}.$$

II. Suppose that $f(z)$ is holomorphic on the real axis and at all but a finite number of points $\{a_1, \ldots, a_n\}$ above the real axis. Let $F(z) = f(z)e^{iz}$. Then integrating $F(z)$ around the piecewise smooth path $\alpha_R + \beta_R$ where $\alpha_R(x) = x$ on $[-R, R]$ and $\beta_R(t) = Re^{it}$ on $[0, \pi]$, we obtain

$$\int_{-R}^{R} f(x)e^{ix}\ dx + \int_{\beta_R} f(z)e^{iz}\ dz = 2\pi i \sum_{k=1}^{n} \text{res}_{a_k} F.$$

See the figure on page 79. Let $||f||_R$ denote the maximum value of $||f(z)||$ on $C_R(0)$. Given that $||e^{iz}|| \le 1$ when z is above the x-axis,

$$\lim_{R\to\infty} R||f||_R = 0 \implies$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)(\cos x + i \sin x)\ dx = 2\pi i \sum_{k=1}^{n} \text{res}_{a_k} F.$$

Example: The function $f(z) = \frac{1}{1+z^2}$ yields

$$\int_0^{\infty} f(x) \cos x\ dx = \pi i \text{res}_i F$$

where $F(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z-i)(z+i)}$. We have

$$\text{res}_i F = \left. \frac{e^{iz}}{z+i} \right|_{z=i} = \frac{e^{-1}}{2i},$$

therefore

$$\int_0^{\infty} \frac{\cos x}{1+x^2}\ dx = \frac{\pi}{2e}.$$

III. We get similar results if $f(z)$ is not holomorphic at $z = 0$, $\lim_{r \to 0} \int_{\beta_r} f(z)\ dz$ exists, and $f(z)$ otherwise meets the conditions above. Just use the indented semicircle on page 105.

Example: Let $a > 0$. To compute $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\ dx$, use

$$f(z) = \frac{\log_{-\pi/2}(z)}{z^2+a^2} = \frac{\log(-iz)}{z^2+a^2} = \frac{\ln r + (\theta - \frac{\pi}{2})i}{z^2+a^2}.$$

When $z = re^{i\theta}$, $a > r > 0$, $0 \leq \theta \leq \pi$, we have

$$r||f||_r \leq \frac{r|\ln r| + r\frac{\pi}{2}}{a^2 - r^2} \to 0 \text{ as } r \to 0.$$

When $z = Re^{i\theta}$, $R > a$, $0 \leq \theta \leq \pi$, we have

$$R||f||_R \leq \frac{R \ln R + R\frac{\pi}{2}}{R^2 - a^2} \to 0 \text{ as } R \to \infty.$$

IV. If $f(z)$ is not holomorphic at a given point along the x-axis, we can try using a semicircular contour that avoids this point. We can apply this method to evaluating $\int_0^{\infty} \frac{1}{1+x^3}\ dx$ – see the exercise set.

Improper Integrals and Rectangular Paths

I. Suppose that $f(z)$ is holomorphic on the real axis and at all but a finite number of points $\{a_1, \ldots, a_n\}$ above the real axis. Assume

$$\forall \epsilon > 0 : \exists R > 0 : ||z| > R \implies ||f(z)|| < \epsilon.$$

Then

$$\int_{-\infty}^{\infty} f(x)e^{ix}\ dx = 2\pi i \cdot \sum_{a_i} \mathrm{res}_{a_i} f(z)e^{iz}.$$

To see that $\int_0^\infty f(x)e^{ix}\ dx$ converges, let M be an upper bound of $||f(z)||$ and choose $T > 0$ so each of the points in $\{a_1, \ldots, a_n\}$ are within T units of the origin. On the path $\alpha(t) = -T + it$, $t \geq 0$, we have $||f(z)e^{iz}|| \leq Me^{-t}$, therefore the integral $\int_0^\infty f(\alpha(t))e^{i\alpha(t)}\alpha'(t)\ dt$ converges. On the path $\beta_q(t) = t + qi$, $t \geq -T$, we have $||f(z)e^{iz}|| \leq Me^{-q}$, which implies that the integral $\int_{-T}^q f(\beta_q(t))e^{-\beta_q(t)}\beta_q'(t)\ dt$ approaches zero as $q \to \infty$. On the path $\gamma_q(t) = q + it$, $t \geq 0$ we have $||f(z)e^{-z}|| \leq M_q$, where M_q is the maximum norm of $f(z)$ on this path, which implies that the integral $\int_0^q f(\gamma_q(t))e^{i\gamma_q(t)}\gamma_q'(t)\ dt$ approaches zero as $q \to \infty$. Integrating around the rectangle with vertices $-T$, q, $q + iq$, $-T + iq$, and letting $q \to \infty$, we obtain

$$\int_{-T}^{\infty} f(x)e^{ix}\ dx = \int_0^\infty f(\alpha(t))e^{i\alpha(t)}\alpha'(t)\ dt + 2\pi \sum_{a_i} \mathrm{res}_{a_i} f(z)e^{iz}.$$

This implies that $\int_0^\infty f(x)e^{ix}\ dx$ converges. Similarly, $\int_{-\infty}^0 f(x)e^{ix}\ dx$ converges.

Integrating around the rectangle with vertices $-R, R, R+Ri, -R+Ri$, and letting $R \to \infty$, we obtain the desired formula.

Example: $f(z) = \frac{z}{z^2+b^2}$ satisfies these conditions and has singularity $z = bi$ above the x-axis. We have

$$\mathrm{res}_{bi} \frac{ze^{iz}}{z^2+b^2} = \left.\frac{ze^{iz}}{z+ib}\right|_{z=ib} = \frac{e^{-b}}{2}.$$

Hence

$$\int_{-R}^0 \frac{xe^{ix}}{x^2+b^2}\ dx + \int_0^R \frac{xe^{ix}}{x^2+b^2}\ dx \to 2\pi i \frac{e^{-b}}{2},$$

$$-\int_R^0 \frac{-ue^{-iu}}{(-u)^2+b^2}\ du + \int_0^R \frac{xe^{ix}}{x^2+b^2}\ dx \to 2\pi i \frac{e^{-b}}{2},$$

$$\int_0^R \frac{2ix\sin x}{x^2+b^2}\ dx \to 2\pi i \frac{e^{-b}}{2},$$
$$\int_0^\infty \frac{x\sin x}{x^2+b^2}\ dx = e^{-b}\frac{\pi}{2}.$$

II. We get similar results if $f(z)$ is not holomorphic at $z=0$, $\lim_{\epsilon\to 0^+}\int_{\beta_r} f(z)e^{iz}\ dz$ exists, and $f(z)$ otherwise meets the conditions above. Use a semicircular indentation about the origin. We obtain

$$\int_{-\infty}^\infty f(x)(\cos x + i\sin x)\ dx + \lim_{\epsilon\to 0^+}\int_{\beta_\epsilon} f(z)e^{iz}\ dz = 2\pi i \sum_{k=1}^n \mathrm{res}_{a_k} F.$$

Example: We can use $f(z)=\frac{1}{z}$ to prove $\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$. This requires the differential approximation

$$\frac{e^{iz}}{z} = \frac{e^{iz}-1}{z} + \frac{1}{z} = i + \psi(z) + \frac{1}{z}$$

where $\psi(z)\to 0$ as $z\to 0$.

Rectangular Paths of Fixed Width

(i) Let $a<b$ and $p<q$ be real numbers. Let $R(a,b,p,q)$ denote the rectangle with sides through $x=a$, $x=b$, $y=p$, $y=q$, and for a function $f(z)$ let $||f||_a$, $||f||_b$, $||f||_p$, and $||f||_q$ denote the maximum value of $||f(z)||$ on each of these sides. Fixing a and b, and assuming that $\lim_{p\to-\infty}||f||_p = 0$ and $\lim_{q\to\infty}||f||_q = 0$ and that for sufficiently large p and q, f is holomorphic on $R(a,b,p,q)$ and has a finite number of singularities in the set S in the interior of $R(a,b,p,q)$, we have

$$i\int_{-\infty}^\infty f(b+it) - f(a+it)\ dt = 2\pi i \sum_{z\in S}\mathrm{res}_z f.$$

(ii) Similarly, fixing p and q, assuming that $\lim_{a\to-\infty}||f||_a = 0$ and $\lim_{b\to\infty}||f||_b = 0$ and that for sufficiently large a and b, f is holomorphic on $R(a,b,p,q)$ and has a finite number of singularities in the set S in the interior of $R(a,b,p,q)$, we have

$$\int_{-\infty}^\infty f(t+ip) - f(t+iq)\ dt = 2\pi i \sum_{z\in S}\mathrm{res}_z f.$$

Example: Let $0 < k < 1$ be given. The function $f(z) = \frac{e^{kz}}{1+e^z}$ meets the conditions in (ii) when $p = 0$ and $q = 2\pi$. This yields

$$\int_{-\infty}^{\infty} \frac{e^{kt}}{1+e^t}\ dt - \int_{-\infty}^{\infty} \frac{e^{kt+2k\pi i}}{1+e^{t+2\pi i}}\ dt = 2\pi i \mathrm{res}_{\pi i} \frac{e^{kz}}{1+e^z}$$

$$(1-e^{2k\pi i}) \int_{-\infty}^{\infty} \frac{e^{kt}}{1+e^t}\ dt = -2\pi i e^{k\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{kt}}{1+e^t}\ dt = \frac{-2\pi i e^{k\pi i}}{1-e^{2k\pi i}} = \frac{\pi}{\sin k\pi}.$$

A Rectilinear Path.

The function $f(z) = \frac{e^{\frac{\pi}{4} i z^2}}{\sin(\frac{\pi}{2} z)}$ is holomorphic on $\mathbb{C} - \{2k : k \in \mathbb{Z}\}$. Let $\alpha = a + bi$ be a non-zero complex number with $a \geq 0$ and $b > 0$. The rectilinear path γ_R around the figure with vertices $1 - R\alpha$, $1 + R\alpha$, $-1 + R\alpha$, $-1 - R\alpha$ encloses the single singularity 0, hence

$$\int_{\gamma_R} f(z)\ dz = 2\pi i \cdot \mathrm{res}_0 \frac{e^{\frac{\pi}{4} i z^2}}{\sin(\frac{\pi}{2} z)} = 2\pi i \cdot \frac{1}{\frac{\pi}{2}} = 4i.$$

The contribution to this integral along the long sides of this path is

$$\alpha \int_{-R}^{R} \frac{e^{\frac{\pi}{4} i (1+\alpha t)^2}}{\sin(\frac{\pi}{2}(1+\alpha t))} + \frac{e^{\frac{\pi}{4} i (1-\alpha t)^2}}{\sin(\frac{\pi}{2}(1-\alpha t))}\ dt =$$

$$\alpha \int_{-R}^{R} \frac{e^{\frac{\pi}{4} i (1+2\alpha t+\alpha^2 t^2)} + e^{\frac{\pi}{4} i (1-2\alpha t+\alpha^2 t^2)}}{\cos(\frac{\pi}{2}\alpha t)}\ dt =$$

$$e^{\frac{\pi}{4} i} \alpha \int_{-R}^{R} \frac{e^{\frac{\pi}{4} i \alpha^2 t^2} \left(e^{\frac{\pi}{2}\alpha t i} + e^{-\frac{\pi}{2}\alpha t i}\right)}{\cos(\frac{\pi}{2}\alpha t)}\ dt =$$

$$4 e^{\frac{\pi}{4} i} \alpha \int_0^R e^{\frac{\pi}{4} i \alpha^2 t^2}\ dt.$$

The contribution along the narrow sides is

$$\alpha \int_{-1}^{1} \frac{e^{\frac{\pi}{4} i (t-\alpha R)^2}}{\sin(\frac{\pi}{2}(t-\alpha R))} - \frac{e^{\frac{\pi}{4} i (t+\alpha R)^2}}{\sin(\frac{\pi}{2}(t+\alpha R))}\ dt = 2\alpha \int_{-1}^{1} \frac{e^{\frac{\pi}{4} i (t-\alpha R)^2}}{\sin(\frac{\pi}{2}(t-\alpha R))}\ dt.$$

Given $||e^{x+iy}|| = e^x$ and $||\sin(x+iy)|| \geq \frac{e^{|y|}}{4}$ for $|y| \geq 1$, we have

$$\left|\left|\frac{e^{\frac{\pi}{4}i(t-\alpha R)^2}}{\sin(\frac{\pi}{2}(t-\alpha R))}\right|\right| \leq e^{\frac{\pi}{2}bR(t-1-aR)},$$

hence

$$\left|\left|\int_{-1}^{1} \frac{e^{\frac{\pi}{4}i(t-\alpha R)^2}}{\sin(\frac{\pi}{2}(t-\alpha R))}\ dt\right|\right| \leq \int_{-1}^{1} e^{\frac{\pi}{2}bR(t-1-aR)}\ dt = \frac{e^{-\frac{\pi}{2}abR}}{\frac{\pi}{2}bR}\left(1-e^{-\pi bR}\right) \to 0$$

as $R \to \infty$. This implies

$$4e^{\frac{\pi}{4}i}\alpha \int_0^\infty e^{\frac{\pi}{4}i\alpha^2 t^2}\ dt = 4i.$$

Rescaling and simplifying,

$$\int_0^\infty e^{se^{i\psi}t^2}\ dt = \sqrt{\frac{\pi}{4s}}e^{(\frac{\pi-\psi}{2})i},$$

$s > 0$ and $\frac{\pi}{2} < \psi \leq \frac{3\pi}{2}$.
Setting $\psi = \pi$ and $s = 1$ we obtain

$$\int_0^\infty e^{-t^2}\ dt = \frac{\sqrt{\pi}}{2}.$$

Setting $\psi = \frac{3\pi}{2}$ and $s = 1$ yields

$$\int_0^\infty \cos(t^2)\ dt = \int_0^\infty \sin(t^2)\ dt = \frac{\sqrt{2\pi}}{4}.$$

Setting $se^{\psi i} = -1 + mi$, $m > 0$, we obtain

$$\int_0^\infty e^{-t^2}\cos(mt^2)\ dt = \frac{\sqrt{\pi}}{4}\left(\sqrt{\frac{\sqrt{m^2+1}+m}{m^2+1}} + \sqrt{\frac{\sqrt{m^2+1}-m}{m^2+1}}\right).$$

Some Infinite Series Evaluations

Theorem: Let $f : \mathbb{C} - S \to \mathbb{C}$ be holomorphic at each $z \in \mathbb{C} - S$, where S is a countable set. For each $n \in \mathbb{N}$ let γ_n denote the piecewise-smooth path parameterizing the square centered at the origin with sides of length $2n+1$. Let I_n an P_n denote the interior and boundary of the square bounded by γ_n. Let $f(z)$ be a function having the following the properties:

1. $S \cap I_n$ is finite for each n.
2. $S \cap B_n = \emptyset$ for each n.
3. There exist real numbers $A > 0$ and $B > 0$ such that $||f(z)|| \leq \frac{A}{||z||^2}$ for all z in the domain of f satisfying $||z|| \geq B$.

Then

$$\lim_{n\to\infty} \sum_{a\in S_n} \text{res}_a f = 0.$$

Proof:

$$\left|\left|\sum_{a\in S_n} \text{res}_a f\right|\right| = \left|\left|\frac{1}{2\pi i}\int_{\gamma_n} f(z)\ dz\right|\right| \leq \frac{A}{2\pi}\frac{8n+4}{(n+\frac{1}{2})^2} \to 0$$

as $n \to \infty$. □

Example: The function $f : \mathbb{C} - \mathbb{Z} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z^6 \sin(\pi z)}$$

is holomorphic at all z in its domain, $S = \mathbb{Z}$, and for each $n \in \mathbb{N}$,

$$S_n = \{-n, -n+1, \ldots, n-1, n\}.$$

Moreover

$$||f(z)||^2 = \frac{1}{||z||^{12}}\frac{1}{||\sin(\pi z)||^2} \leq \frac{16}{||z||^{12}} \leq \frac{16}{||z||^4}$$

for all z in the domain of f satisfying $||z|| \geq 1$. Hence f meets the hypotheses of the theorem. Hence

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \text{res}_k f = \lim_{n\to\infty} \sum_{k=-n}^{n} \text{res}_k \frac{1}{z^6 \sin(\pi z)} = 0.$$

When $k = 0$ we have

$$\text{res}_0 \frac{1}{z^6 \sin(\pi z)} =$$

$$\mathrm{res}_0 \frac{1}{z^6}\left(\frac{1}{\pi z} + \frac{1}{6}\pi z + \frac{7}{360}\pi^3 z^3 + \frac{31}{15120}\pi^5 z^5 + \cdots\right) = \frac{31\pi^5}{15120}.$$

When $k \neq 0$,

$$\mathrm{res}_k \frac{1}{z^6}\frac{1}{\sin(\pi(z-k)+\pi k)} =$$

$$\mathrm{res}_k \frac{1}{z^6}\frac{(-1)^k}{\sin(\pi(z-k))} =$$

$$\mathrm{res}_k \frac{1}{z-k}\frac{1}{\pi z^6}\frac{(-1)^k}{\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{(2n+1)!}(z-k)^{2n}} = \frac{(-1)^k}{\pi k^6}.$$

Therefore

$$\frac{31\pi^5}{15120} + \lim_{n\to\infty}\frac{2}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{k^6} = 0,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^6} = \frac{31\pi^6}{30240}.$$

Example: The function $f : \mathbb{C} - \mathbb{Z} \to \mathbb{C}$ defined by

$$f(z) = \frac{\cos(\pi z)}{z^6 \sin(\pi z)}$$

is holomorphic at all z in its domain, $S = \mathbb{Z}$, and for each $n \in \mathbb{N}$,

$$S_n = \{-n, -n+1, \ldots, n-1, n\}.$$

Moreover

$$||f(z)||^2 = \frac{||\cot^2(\pi z)||}{||z||^{12}} = \frac{||\csc^2(\pi z)+1||}{||z||^{12}} \leq \frac{17}{||z||^{12}} \leq \frac{17}{||z||^4}$$

for all z in the domain of f satisfying $||z|| \geq 1$. Hence f meets the hypotheses of the theorem. Hence

$$\lim_{n\to\infty}\sum_{k=-n}^{n}\mathrm{res}_k f = \lim_{n\to\infty}\sum_{k=-n}^{n}\mathrm{res}_k \frac{\cos(\pi z)}{z^6\sin(\pi z)} = 0.$$

When $k = 0$ we have

$$\mathrm{res}_0 \frac{\cos(\pi z)}{z^6 \sin(\pi z)} =$$

$$\operatorname{res}_0 \frac{1}{z^6}\left(\frac{1}{\pi z} - \frac{1}{3}\pi z - \frac{1}{45}\pi^3 z^3 - \frac{2}{945}\pi^5 z^5 - \cdots\right) = -\frac{2\pi^5}{945}.$$

When $k \neq 0$,

$$\operatorname{res}_k \frac{\cos(\pi z)}{z^6 \sin(\pi z)} = \operatorname{res}_k \frac{1}{z^6} \frac{\cos(\pi(z-k)+\pi k)}{\sin(\pi(z-k)+\pi k)} =$$

$$\operatorname{res}_k \frac{1}{z^6} \frac{\cos(\pi(z-k))}{\sin(\pi(z-k))} =$$

$$\operatorname{res}_k \frac{1}{z-k} \frac{1}{\pi z^6} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}(z-k)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}(z-k)^{2n}} = \frac{1}{\pi k^6}.$$

Therefore

$$-\frac{2\pi^5}{945} + \lim_{n\to\infty} \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k^6} = 0,$$

$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}.$$

Analytic Continuation of Holomorphic Functions

Definition: Let $f : S \to \mathbb{C}$ be holomorphic on S. If $S \subseteq T$ and $F : T \to \mathbb{C}$ is holomorphic on T and satisfies $F(z) = f(z)$ for all $z \in S$, then we say that F is an analytic continuation of f to the set T.

Example: Let $f : B_r(a) - \{a\}$ have Laurent series expansion

$$f(z) = \sum_{n=-1}^{\infty} c_n (z-a)^n.$$

Then $f(z) - \frac{c_{-1}}{z-a}$ has analytic continuation $\sum_{n=0}^{\infty} c_n (z-a)^n$ to $B_r(a)$.

The Riemann Zeta Function

I. Definition of the Riemann Zeta Function

Recall that for a complex number $z \in \mathbb{C} - \{x + 0i : x < 0\}$ and for any other complex number w, $z^w = e^{w \log z}$. In particular, for a positive integer n, $n^{x+iy} = e^{(x+iy)\log n} = n^x \cos(n^y) + n^x \sin(n^y) i$.

Definition: The Riemann Zeta function is the function

$$\zeta : \{z \in \mathbb{C} : \text{re } z > 1\} \to \mathbb{C}$$

defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Since $||n^{x+iy}|| = n^x$, $\zeta(z)$ is absolutely convergent for each z in its domain.

Lemma: Let S be an open and convex subset of $\mathbb{C}$ and for each $n \geq 0$ let $f_n : S \to \mathbb{C}$ be holomorphic on S. If $f_n \to f$ uniformly on S then f is holomorphic on S and $f'(z) = \lim_{n\to\infty} f_n'(z)$ for each $z \in S$.

Proof: Since each f_n is continuous, f is continuous on S. Moreover, for any piecewise smooth γ_T parameterizing a triangle T in S,

$$\int_{\gamma_T} f(z)\ dz = \lim_{n\to\infty} \int_{\gamma_T} f_n(z)\ dz = 0$$

since each $f_n(z)$ is holomorphic on S. Therefore f has an antiderivative F on S by Morera's Theorem. Since F is infinitely differentiable on S, so is f.

Now let $z \in S$ be given. Choose $r > 0$ so that $C_r(z) \subseteq S$. By Cauchy's Integral Formula,

$$f_n'(z) - f'(z) = \frac{2!}{2\pi i} \int_{C_r(z)} \frac{f_n(w) - f(w)}{(w-z)^2}\ dz,$$

therefore by the M-L inequality we have

$$||f_n'(z) - f'(z)|| \leq \left|\left| \frac{2!}{2\pi i} \int_{C_r(z)} \frac{f_n(w) - f(w)}{(w-z)^2}\ dz \right|\right| \leq \frac{1}{\pi} 2\pi r \frac{||f_n - f||}{r^2} \to 0$$

as $n \to \infty$. $\square$

Corollary: Let S be an open and convex subset of $\mathbb{C}$ and for each $n \geq 0$ let $f_n : S \to \mathbb{C}$ be holomorphic on S. If $\sum_{n=0}^{\infty} ||f_n||$ converges then $\sum_{n=0}^{\infty} f_n$ is holomorphic and has derivative equal to $\sum_{n=0}^{\infty} f_n'$. $\square$

Theorem: The Riemann Zeta function is holomorphic at each z in its domain.

Proof: Let $z_0 = x_0 + iy_0$ be given, where $x_0 > 1$. Fix x_1 satisfying $1 < x_1 < x_0$, and set

$$X_1 = \{x + iy \in \mathbb{C} : x > x_1\}.$$

For each $n \in \mathbb{N}$ define $f_n : X_1 \to \mathbb{C}$ by

$$f_n(z) = \frac{1}{n^z}.$$

Then each f_n is holomorphic on X_1 and we have

$$\sum_{n=1}^{\infty} ||f_n|| = \sum_{n=1}^{\infty} \frac{1}{n^{x_1}} < \infty.$$

Hence ζ is on X_1, and in particular at z_0. Moreover

$$\zeta'(z) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^z}.$$

$\square$

II. The Euler Product Formula

Lemma: Let (p_n) be the sequence of prime numbers, let $\mathbb{N}_0 = \mathbb{N}$, and for $k \geq 0$ let

$$\mathbb{N}_k = \{n \in \mathbb{N} : n \text{ is not divisible by } p_i \text{ for } 1 \leq i \leq k\}.$$

Then for all k,

$$\mathbb{N}_{k+1} = \mathbb{N}_k - \{p_{k+1}n : n \in \mathbb{N}_k\}.$$

Proof: It is clear that

$$\mathbb{N}_{k+1} \subseteq \mathbb{N}_k - \{p_{k+1}n : n \in \mathbb{N}_k\}.$$

Now let $x \in \mathbb{N}_k - \{p_{k+1}n : n \in \mathbb{N}_k\}$ be given. Then x is not divisible by any of the primes $p_1, \ldots, p_k$, and so $x = p_{k+1}^r m$ for some $r \geq 0$ and $m \in \mathbb{N}_{k+1}$. If $r > 0$ then $x = p_{k+1}n$ where $n = p_{k+1}^{r-1}m \in \mathbb{N}_k$, a contradiction. Therefore $r = 0$ and $x = m \in \mathbb{N}_{k+1}$. $\square$

Lemma: Fix a real number $x_0 > 1$. Then

$$\zeta(z)\prod_{k=1}^{n}\left(1-\frac{1}{p_n^z}\right)\to 1$$

uniformly on $\{x+iy\in\mathbb{C} : x\geq x_0\}$.

Proof: For any $k\geq 0$ we have

$$\sum_{n\in\mathbb{N}_k}\frac{1}{n^z}\left(1-\frac{1}{p_{k+1}^z}\right)=\sum_{n\in\mathbb{N}_k}\frac{1}{n^z}-\sum_{n\in\mathbb{N}_k}\frac{1}{(p_{k+1}n)^z}=\sum_{n\in\mathbb{N}_{k+1}}\frac{1}{n^z}$$

by the Lemma. This yields the sequence of identities

$$\zeta(z)\left(1-\frac{1}{p_1^z}\right)=\sum_{n\in\mathbb{N}_1}\frac{1}{n^z},$$

$$\zeta(z)\left(1-\frac{1}{p_1^z}\right)\left(1-\frac{1}{p_2^z}\right)=\sum_{n\in\mathbb{N}_2}\frac{1}{n^z},$$

$$\zeta(z)\left(1-\frac{1}{p_1^z}\right)\left(1-\frac{1}{p_2^z}\right)\left(1-\frac{1}{p_3^z}\right)=\sum_{n\in\mathbb{N}_3}\frac{1}{n^z},$$

etc. Since $\mathbb{N}_k=\{1\}\cup S_k$ where $S_k\subseteq\{p_k+1,p_k+2,\dots\}$,

$$\left\|\zeta(z)\prod_{k=1}^{n}\left(1-\frac{1}{p_n^z}\right)-1\right\|\leq\sum_{n=p_k+1}^{\infty}\frac{1}{n^{x_0}}\to 0$$

as $k\to\infty$. □

Corollary: For all $z\in\mathbb{C}$ with re $z>1$,

$$\zeta(z)=\prod_{n=1}^{\infty}\frac{1}{1-\frac{1}{p_n^z}}.$$

III. The Logarithmic Derivative of $\zeta(z)$

Theorem: For all $z\in\mathbb{C}$ satisfying re $z>1$,

$$\frac{\zeta'(z)}{\zeta(z)}=\sum_{n=1}^{\infty}\frac{\log p_n}{1-p_n^z}.$$

Proof: Write

$$\Pi_n(z) = \prod_{k=1}^{n} \left(1 - \frac{1}{p_n^z}\right).$$

Fix $z_0 = x_0 + iy_0$ with $x_0 > 1$. Choose x_1 satisfying $1 < x_1 < x_0$. By uniform convergence on $\{x + iy : x > x_1\}$ and the lemma in **I**, for all z in this set we have

$$(\zeta(z)\Pi_n(z))' \to 0.$$

Hence

$$\zeta'(z)\Pi_n(z) + \zeta(z)\Pi_n'(z) \to 0,$$

$$\frac{\zeta'(z)}{\zeta(z)} \to \frac{\Pi_n'(z)}{\Pi_n(z)},$$

$$\frac{\zeta'(z)}{\zeta(z)} = -\lim_{n\to\infty} \frac{\Pi_n'(z)}{\Pi_n(z)} = -\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(1-p_k^{-z})'}{1-p_k^{-z}} = \sum_{n=1}^{\infty} \frac{\log p_n}{1-p_n^z}.$$

□

IV. Analytic Continuation of $\zeta(z)$ to $\{z \in \mathbb{C} : \text{re } z > 0\} - \{1\}$

Lemma: For all $z \in \mathbb{C}$ with re $z > 1$,

$$\frac{1}{z-1} = \int_1^\infty \frac{1}{x^z}\, dx.$$

Proof: Fix $z = a + bi$ where $a > 1$. Making the change of variables $u = \ln x$, we have

$$\int_1^R \frac{1}{x^z}\, dx = \int_0^{\ln R} e^{u(1-z)}\, du = \left.\frac{e^{u(1-z)}}{1-z}\right|_0^{\ln R} = \frac{e^{\ln R(1-z)} - 1}{z-1}.$$

Since $a > 1$,

$$\left|\left|e^{\ln R(1-z)}\right|\right| = \left|\left|e^{\ln R - a\ln R - b\ln Ri}\right|\right| = e^{(1-a)\ln R} \to 0$$

as $R \to \infty$ since $1 - a < 0$. This implies

$$\int_1^\infty \frac{1}{x^z}\, dx = \frac{1}{z-1}.$$

$\square$

Theorem: The function $F : \{z \in \mathbb{C} : \text{re } z > 0\} \to \mathbb{C}$ defined by

$$F(z) = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z} \right) \, dx$$

is holomorphic on its domain and satisfies $F(z) = \zeta(z) - \frac{1}{z-1}$ for all $z \in \mathbb{C}$ such that re $z > 1$.

Proof: Fix $z = a + bi$ with $a > 1$. Then

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{1}{x^z} \, dx = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z} \right) \, dx.$$

By a previous exercise, for each $n \geq 1$ the function $f_n : B_{\frac{a}{2}}(a + bi) \to \mathbb{C}$ defined by

$$f_n(z) = \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z} \right) \, dx$$

is the uniform limit of

$$\sum_{k=0}^{\infty} \int_{n}^{n+1} \frac{(\ln n)^k - (\ln x)^k}{k!} (-z)^k \, dx.$$

Since each summand in the latter expression is a holomorphic function of z on $B_{\frac{a}{2}}(a + bi)$, $f_n(z)$ is holomorphic on $B_{\frac{a}{2}}(a + bi)$. By a previous exercise, $\sum_{n=0}^{\infty} f_n$ converges uniformly on $B_{\frac{a}{2}}(a + bi)$, hence is holomorphic on that set. Since

$$\{z \in \mathbb{C} : \text{re } z > 0\} = \bigcup_{(a,b) \in (0,\infty) \times \mathbb{R}} B_{\frac{a}{2}}(a + bi),$$

$\sum_{n=0}^{\infty} f_n$ is holomorphic on $\{z \in \mathbb{C} : \text{re } z > 0\}$. $\square$

We will define $\zeta_1(z) = F(z) + \frac{1}{z-1}$ for all $z \in \{z \in \mathbb{C} : \text{re } z > 0\} - \{1\}$. Since both $F(z)$ and $\frac{1}{z-1}$ are holomorphic in this domain, so is $\zeta_1(z)$. We have $\zeta_1(z) = \zeta(z)$ for all z in the domain of ζ.

V. $\zeta_1(z)$ has no zeros on the line re $z = 1$

Theorem: For all $z \in \mathbb{C}$ with re $z = 1$ and $z \neq 1$, $\zeta_1(z) \neq 0$.

Proof: Note that for any $a + bi$ with $a > 1$ and prime number p we have

$$\left|\left| p^{a+bi} \right|\right| = p^a > 1,$$

therefore

$$1 - \frac{1}{p^{a+bi}} \in B_1(1+0i),$$

therefore $\log(1 - \frac{1}{p^{a+bi}})$ is defined. Hence

$$||\zeta_1(a+bi)|| \cdot ||\Pi_n(a+bi)|| \to 1,$$

$$\ln||\zeta_1(a+bi)|| + \ln||\Pi_n(a+bi)|| \to 0,$$

$$\ln||\zeta_1(a+bi)|| + \sum_{k=1}^{n} \ln\left|\left|\left(1 - \frac{1}{p_k^{a+bi}}\right)\right|\right| \to 0,$$

$$\ln||\zeta_1(a+bi)|| = -\sum_{k=1}^{\infty} \ln\left|\left|\left(1 - \frac{1}{p_k^{a+bi}}\right)\right|\right| =$$

$$-\mathrm{re}\ \sum_{k=1}^{\infty} \log\left(1 - \frac{1}{p_k^{a+bi}}\right) =$$

$$\mathrm{re}\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{p_k^{-(a+bi)n}}{n} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{\cos(\ln p^{bn})}{np_k^{an}},$$

hence

$$\ln||\zeta_1(a)^3\zeta_1(a+bi)^4\zeta_1(a+2bi)|| =$$

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{3 + 4\cos(\ln p^{bn}) + \cos(\ln p^{2bn})}{np_k^{an}}.$$

Each summand in this expression is non-negative: setting $\theta_n = \ln(p^{bn})$ we have

$$3 + 4\cos(\ln p^{bn}) + \cos(\ln p^{2bn}) = 3 + 4\cos(\theta_n) + \cos(2\theta_n) = 2(\cos\theta_n + 1)^2 \geq 0.$$

This implies

$$||\zeta_1(a)^3\zeta_1(a+bi)^4\zeta_1(a+2bi)|| \geq 1.$$

Now suppose that $\zeta_1(1+bi) = 0$ for some $b \neq 0$. Since F is continuous at $z = 1$ and $\zeta_1(a) = F(a) + \frac{1}{a-1}$ for all $a > 1$,

$$\lim_{a \to 1^+} (a-1)\zeta_1(a) = 1.$$

Since ζ_1 is holomorphic at $1+bi$,

$$\lim_{a\to 1^+} \frac{\zeta_1(a+bi)}{a-1} = \lim_{a\to 1^+} \frac{\zeta_1(a+bi)-\zeta_1(1+bi)}{a-1} = \zeta_1'(1+bi).$$

Therefore

$$\lim_{a\to 1^+} \frac{\zeta_1(a)^3\zeta_1(a+bi)^4\zeta_1(a+2bi)}{a-1} = \lim_{a\to 1^+} (a-1)^3\zeta_1(a)^3 \cdot \frac{\zeta_1(a+bi)^4}{(a-1)^4} \cdot \zeta_1(a+2bi) =$$

$$\zeta_1'(1+bi)^4\zeta_1(1+2bi).$$

But this contradicts

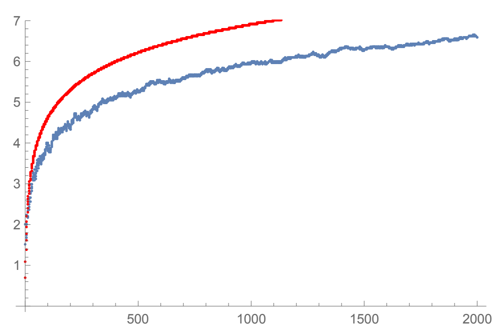
$$\left\|\frac{\zeta_1(a)^3\zeta_1(a+bi)^4\zeta_1(a+2bi)}{a-1}\right\| \geq \frac{1}{a-1}$$

for all $a > 1$. Therefore $\zeta_1(1+bi) \neq 0$ for all $b \neq 0$.

<u>The Prime Number Theorem</u>

Definition: Let $n \geq 2$ be a real number. Then $\pi(n)$ is the number of prime numbers $\leq n$.

Remark: If we name the primes $p_1, p_2, p_3, \ldots$ in increasing order, then the larger n is, the more ways there are to form products of p_1 through $p_{\pi(n)}$ yielding all the numbers in $\{1, 2, \ldots, n\}$. One would expect that $\frac{\pi(n)}{n} \to 0$ as $n \to \infty$, or equivalently $\frac{n}{\pi(n)} \to \infty$. A graph of $\frac{n}{\pi(n)}$ versus n resembles the graph of $\log n$ versus n:



Prime Number Theorem:

$$\lim_{n\to\infty} \frac{\pi(n)\log(n)}{n} = 1.$$

We will prove this theorem in stages below.

Tchebychev's Theta Function

Definition: The Tchebychev Theta Function is defined by

$$\theta(x) = \sum_{p \le x} \log p,$$

the sum ranging over prime numbers bounded above by x.

Theorem: For all $x \ge 1$, $\theta(x) < x \log 16$.

Proof: For any $k \in \mathbb{N}$, all the prime numbers between $2^{k-1}+1$ and 2^k divide the binomial coefficient $\binom{2^k}{2^{k-1}}$, hence

$$\left(\prod_{2^{k-1} < p \le 2^k} p \right) \Bigg| \binom{2^k}{2^{k-1}},$$

hence

$$\prod_{2^{k-1} < p \le 2^k} p \le \binom{2^k}{2^{k-1}} \le 2^{2^k}.$$

This implies

$$\prod_{1 < p \le 2^k} p \le 2^{2^1+2^2+\cdots+2^k} = 2^{2^{k+1}-2},$$

$$\theta(2^k) \le (2^{k+1} - 2) \log 2.$$

Given $x \ge 1$, choose $k \in \mathbb{N}$ such that $2^{k-1} \le x < 2^k$. Then

$$\theta(x) \le \theta(2^k) < 2^{k+1} \log 2 = 4 \cdot 2^{k-1} \log 2 \le x \log 16.$$

Theorem:

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1 \implies \lim_{x \to \infty} \frac{\pi(x) \log(x)}{x} = 1.$$

Proof: Assume $\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$. Let $\epsilon > 0$ be given. Write $\delta = \frac{1}{1+\frac{\epsilon}{4}}$. We have

$$\theta(x) = \sum_{p \le x} \log p \le \pi(x) \log x$$

and

$$\theta(x) \geq \sum_{x^\delta < p \leq x} \log p \geq (\pi(x) - \pi(x^\delta)) \log(x^\delta) \geq (\pi(x) - x^\delta)\delta \log x,$$

therefore

$$\frac{\theta(x)}{x} \leq \frac{\pi(x)\log(x)}{x} \leq \left(1 + \frac{\epsilon}{4}\right) \frac{\theta(x)}{x} + \frac{\log x}{x^{1-\delta}}.$$

For x sufficiently large we have

$$1 - \epsilon < \frac{\theta(x)}{x} < \frac{1 + \frac{\epsilon}{2}}{1 + \frac{\epsilon}{4}}$$

and

$$\frac{\log x}{x^{1-\delta}} < \frac{\epsilon}{2},$$

hence

$$1 - \epsilon < \frac{\pi(x)\log(x)}{x} < 1 + \epsilon. \quad \square$$

A Condition that Implies $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$

Theorem: If the improper integral

$$\int_0^\infty \theta(e^t)e^{-t} - 1 \; dt$$

converges then $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$.

Proof: Assume that

$$\int_0^\infty \theta(e^t)e^{-t} - 1 \; dt$$

converges. Making the change of variables $x = e^t$, the improper integral

$$\int_1^\infty \frac{1}{x}\left(\frac{\theta(x)}{x} - 1\right) \; dx$$

converges. Suppose $\lim_{x\to\infty} \frac{\theta(x)}{x} \neq 1$. Then there exists $\epsilon > 0$ and a sequence (x_n) such that $x_n \geq n$ and

$$\left|\frac{\theta(x_n)}{x_n} - 1\right| \geq \epsilon$$

for each n. Hence either $\frac{\theta(x_n)}{x_n} - 1 \geq \epsilon$ for infinitely many n or $\frac{\theta(x_n)}{x_n} - 1 \leq -\epsilon$ for infinitely many n. The two cases are similar, so we will just treat the first case.

Choose a subsequence (y_n) of (x_n) satisfying $\frac{\theta(y_n)}{y_n} - 1 \geq \epsilon$ and $y_{n+1} \geq (1+\epsilon)y_n$ for each n. For each n we have

$$\int_{y_n}^{(1+\epsilon)y_n} \frac{1}{x}\left(\frac{\theta(x)}{x} - 1\right) \; dx \geq \int_{y_n}^{(1+\epsilon)y_n} \frac{1}{x}\left(\frac{\theta(y_n)}{x} - 1\right) \; dx \geq$$

$$\int_{y_n}^{(1+\epsilon)y_n} \frac{1}{x}\left(\frac{(1+\epsilon)y_n}{x} - 1\right) \; dx = \epsilon - \log(1+\epsilon) > 0.$$

This implies

$$\int_1^{y_n} \frac{1}{x}\left(\frac{\theta(x)}{x} - 1\right) \; dx \geq n(\epsilon - \log(1+\epsilon)) \to \infty$$

as $n \to \infty$, a contradiction. Therefore $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$. $\square$

__**The Laplace Transform of $\theta(e^t)e^{-t} - 1$**__

Theorem: For all $z \in \mathbb{C}$ with re $z > 0$,

$$\int_0^\infty (\theta(e^t)e^{-t} - 1)e^{-tz}\ dt = \frac{1}{z+1}\sum_{n=1}^\infty \frac{\log p_n}{p_n^{z+1}} - \frac{1}{z}.$$

Proof: Let $z = \sigma + \tau i \in \mathbb{C}$ with $\sigma > 1$ be given. For each $k \in \mathbb{N}$ let

$$\gamma_k : [\log p_k, \log_{p_{k+1}}] \to \mathbb{C}$$

be defined by

$$\gamma_k(t) = \theta(e^t)e^{-tz}.$$

Then for $\log p_k \leq t < \log p_{k+1}$,

$$\gamma_k(t) = \theta(p_k)e^{-tz}.$$

Given that an antiderivative for e^{-tz} with respect to t is $\frac{-1}{z}e^{-tz}$, we have

$$\int_{\log p_k}^{\log p_{k+1}} \gamma_n(t)\ dt = \frac{-1}{z}\gamma(t)\Big|_{\log p_k}^{\log p_{k+1}} = \frac{\theta(p_k)}{z}\left(\frac{1}{p_k^z} - \frac{1}{p_{k+1}^z}\right).$$

This implies

$$\int_0^{\log p_{n+1}} \theta(e^t)e^{-tz}\ dt = \frac{1}{z}\sum_{k=1}^n \theta(p_k)\left(\frac{1}{p_k^z} - \frac{1}{p_{k+1}^z}\right) =$$

$$\frac{1}{z}\sum_{k=1}^n\sum_{i=1}^k \log p_i\left(\frac{1}{p_k^z} - \frac{1}{p_{k+1}^z}\right) = \frac{1}{z}\sum_{i=1}^n \log p_i \sum_{k=i}^n\left(\frac{1}{p_k^z} - \frac{1}{p_{k+1}^z}\right) =$$

$$\frac{1}{z}\sum_{i=1}^n \log p_i\left(\frac{1}{p_i^z} - \frac{1}{p_{n+1}^z}\right) = \left(\frac{1}{z}\sum_{i=1}^n \frac{\log p_i}{p_i^z}\right) - \frac{\theta(p_n)}{p_{n+1}^z}.$$

Given that

$$\left\|\frac{\theta(p_n)}{p_{n+1}^z}\right\| < \frac{\log 16}{p_n^{\sigma-1}} \to 0$$

as $n \to \infty$,

$$\int_0^\infty \theta(e^t)e^{-tz}\ dt = \frac{1}{z}\sum_{i=1}^\infty \frac{\log p_i}{p_i^z}.$$

Hence for $z \in \mathbb{C}$ with re $z > 0$,

$$\int_0^\infty \theta(e^t)e^{-t}e^{-tz}\ dt = \int_0^\infty \theta(e^t)e^{-t(z+1)}\ dt = \frac{1}{z+1}\sum_{i=1}^\infty \frac{\log p_i}{p_i^{z+1}}.$$

Combining this with

$$\int_0^\infty e^{-tz}\ dt = \frac{1}{z}$$

completes the proof. □

Analytic Continuation of $\int_0^\infty (\theta(e^t)e^{-t} - 1)e^{-tz}\ dt$

The expression $\sum_{n=1}^\infty \frac{\log p_n}{p_n^{z+1}}$ converges for each $z \in \mathbb{C}$ with re $z > 0$, and we have

$$\sum_{n=1}^\infty \frac{\log p_n}{p_n^{z+1}} = \sum_{n=1}^\infty \frac{\log p_n}{p_n^{z+1} - 1} - \sum_{n=1}^\infty \frac{\log p_n}{p_n^{2z+2} - p_n^{z+1}}.$$

The expression

$$\sum_{n=1}^\infty \frac{\log p_n}{p_n^{2z+2} - p_n^{z+1}}$$

is holomorphic at all $z \in \mathbb{C}$ satisfying re $z > -\frac{1}{2}$. For all $z \in \mathbb{C}$ satisfying re $z > 0$ we have

$$\zeta_1'(z+1) = \zeta_1(z+1)\sum_{n=1}^\infty \frac{\log p_n}{1 - p_n^{z+1}}.$$

Since $\zeta_1(z+1)$ is holomorphic for all $z \in \mathbb{C}$ satisfying re $z > -1$ and $z \neq 0$, and is non-zero when re $(z) \geq 0$, the expression

$$-\frac{\zeta_1'(z+1)}{\zeta_1(z+1)} = -\frac{F'(z+1) - \frac{1}{z^2}}{F(z+1) + \frac{1}{z}} = \frac{-z^2F'(z+1) + 1}{z^2F(z+1) + z}$$

is holomorphic at each $z \neq 0$ satisfying re $z \geq 0$ and agrees with

$$\sum_{n=1}^\infty \frac{\log p_n}{p_n^{z+1} - 1}$$

when re $z > 0$. Hence an analytic continuation of $\sum_p \frac{\log p}{p^{z+1}}$ to all $z \neq 0$ satisfying re $z \geq 0$ is

$$G(z) = \frac{-z^2F'(z+1) + 1}{z^2F(z+1) + z} - \sum_{n=1}^\infty \frac{\log p_n}{p_n^{2z+2} - p_n^{z+1}}.$$

Hence

$$H(z) = \frac{1}{z+1}G(z) - \frac{1}{z} =$$

$$\frac{-z^2F'(z+1)+1}{(z+1)(z^2F(z+1)+z)} - \frac{1}{z+1}\sum_{n=1}^{\infty}\frac{\log p_n}{p_n^{2z+2}-p_n^{z+1}} - \frac{1}{z} =$$

$$\frac{-zF'(z+1)-(z+1)F(z+1)-1}{(z+1)(zF(z+1)+1)} - \frac{1}{z+1}\sum_{n=1}^{\infty}\frac{\log p_n}{p_n^{2z+2}-p_n^{z+1}}$$

is an analytic continuation of

$$\frac{1}{z+1}\sum_p \frac{\log p}{p^{z+1}} - \frac{1}{z}$$

on this set. Since the Laurent series expansion of $H(z)$ does not include any negative powers of z, the resulting power series expansion represents an analytic continuation of $\int_0^\infty (\theta(e^t)e^{-t}-1)e^{-tz}\ dt = \sum_p \frac{\log p}{p^{z+1}}$ to the set $\{z \in \mathbb{C} : \text{re }(z) \geq 0\}$. This has a constant term of

$$I(0) = -F(1) - 1 - \sum_{n=1} \frac{\log p_n}{p_n^2 - p_n}.$$

Proof that $\int_0^\infty (\theta(e^t)e^{-t}-1)e^{-tz}\ dt$ converges

Let $I(z)$ be the analytic continuation of $\int_0^\infty (\theta(e^t)e^{-t}-1)e^{-tz}\ dt$ to a neighborhood of $\{z \in \mathbb{C} : \text{re }(z) \geq 0\}$. While we have proved that the integral expression converges for all z satisfying re $z > 0$, we do not yet know that it converges using $z = 0$.

Theorem: $\int_0^\infty \theta(e^t)e^{-t} - 1\ dt = I(0)$.

Proof: For each $T > 0$ define $g_T : \mathbb{C} \to \mathbb{C}$ by

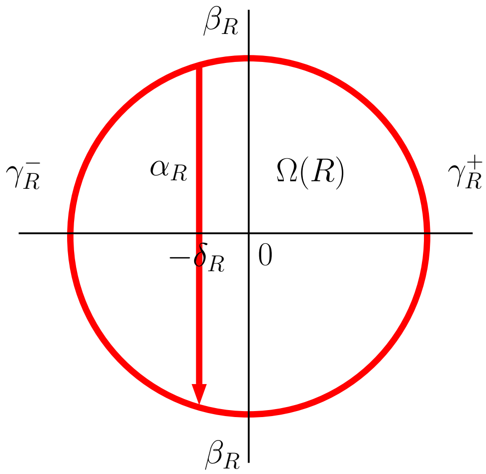
$$g_T(z) = \int_0^T (\theta(e^t)e^{-t}-1)e^{-zt}\ dt.$$

We can verify, in the usual way, that each g_T is holomorphic.

Fix R. For each $y \in [-R, R]$ there is $\epsilon(y) > 0$ such that $I(z)$ is holomorphic on $B_{\epsilon(y)}(0+bi)$. By a compactness argument there exists $\delta_R > 0$ such that $I(z)$ is holomorphic on

$$\{x + yi : (x, y) \in [-\delta_R, \infty) \times [-R, R]\}.$$

Let C_R be the counterclockwise path in this set that winds around the origin and incorporates the circle of radius R about the origin and the line re $(z) = -\delta_R$. We will write $C_R = \alpha_R + \beta_R + \gamma_R^+$, where α_R is the vertical part of C_R, β_R is the circular part of C_R where re $(z) \le 0$, and γ_R^+ is the circular part of C_R where re $(z) \ge 0$. We will also denote by $\Omega(R)$ the region bounded by C_R and γ_R^- the counterclockwise path around the semicircle $||z|| = R$ where re $(z) \le 0$.



By Cauchy's Integral Formula we have

$$I(0) - g_T(0) = \frac{1}{2\pi i}\int_{C_R} (I(z) - g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z},$$

the extra factor of $e^{Tz}(1 + \frac{R^2}{z^2})$ included to simplify some of the calculations. Observe that for $||z|| = R$ and $z = x + iy$ we have

$$\left|\left|1 + \frac{z^2}{R^2}\right|\right| = \left|\left|\frac{z\overline{z} + z^2}{R^2}\right|\right| = \frac{||z||}{R^2}\,|\overline{z} + z| = \frac{2|x|}{R}.$$

For $z \in \gamma_R^+$ and $z = x + iy$ and $x > 0$ we have

$$||I(z) - g_T(z)|| = \left|\left|\int_T^\infty (\theta(e^t)e^{-t} - 1)e^{-zt}\ dt\right|\right| \le$$

$$\int_T^\infty \left|\left|(\theta(e^t)e^{-t} - 1)e^{-zt}\right|\right|\ dt \le 17\int_T^\infty e^{-xt}\ dt = \frac{17e^{-xT}}{x},$$

therefore

$$\left|\left|(I(z) - g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{1}{z}\right|\right| \leq \frac{17e^{-xT}}{x}e^{xT}\frac{2x}{R}\frac{1}{R} = \frac{34}{R^2},$$

therefore

$$\left|\left|\frac{1}{2\pi i}\int_{\gamma_R^+}(I(z) - g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z}\right|\right| \leq \frac{17}{R}.$$

Since $g_T(z)$ is entire,

$$\int_{\alpha_R+\beta_R}(g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z} = \int_{\gamma_R^-}(g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z}.$$

For $z \in \gamma_R^-$ and $z = -x + iy$ and $x > 0$ we have

$$||g_T(z)|| = \left|\left|\int_0^T (\theta(e^t)e^{-t} - 1)e^{-zt}\ dt\right|\right| \leq$$

$$\int_0^T \left|\left|(\theta(e^t)e^{-t} - 1)e^{-zt}\right|\right|\ dt \leq 17\int_0^T e^{-xt}\ dt \leq \frac{17}{x},$$

therefore

$$\left|\left|g_T(z)e^{Tz}(1 + \frac{z^2}{R^2})\frac{1}{z}\right|\right| \leq \frac{17}{x}e^{-xT}\frac{2x}{R}\frac{1}{R} \leq \frac{34}{R^2},$$

therefore

$$\left|\left|\frac{1}{2\pi i}\int_{\alpha_R+\beta_R}g_T(z)e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z}\right|\right| = \left|\left|\frac{1}{2\pi i}\int_{\gamma_R^-}g_T(z)e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z}\right|\right| \leq \frac{17}{R}.$$

Since $I(z)$ is continuous on $\Omega(R)$ and $\Omega(R)$ is compact,

$$\sup\{||I(z)|| : z \in \Omega(R)\} = ||I(z(R, \delta_R))||$$

for some $z(R, \delta_R) \in C_R$. This yields

$$\left|\left|\frac{1}{2\pi i}\int_{\alpha_R}I(z)e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z}\right|\right| \leq \frac{1}{2\pi}\cdot||I(z(R, \delta_R))||\cdot e^{-\delta_R T}\cdot 2\cdot\frac{1}{\delta_R}\cdot 2R =$$

$$\frac{2||I(z(R, \delta_R))||e^{-\delta_R T}R}{\pi\delta_R}.$$

Given that the length of β_R is $2R\sin^{-1}\frac{\delta_R}{R}$, we have

$$\left|\left|\frac{1}{2\pi i}\int_{\beta_R} I(z)e^{Tz}(1+\frac{z^2}{R^2})\frac{dz}{z}\right|\right| \leq \frac{1}{2\pi}\cdot||I(z(R,\delta_R))||\cdot 1\cdot 2\cdot\frac{1}{R}\cdot 2R\sin^{-1}\frac{\delta_R}{R} =$$

$$\frac{2}{\pi}||I(z(R,\delta_R))||\sin^{-1}\frac{\delta_R}{R}.$$

Therefore

$$||I(0)-g_T(0)|| \leq \frac{34}{R}+\frac{2}{\pi}||I(z(R,\delta_R))||\left(\frac{e^{-\delta_R T}R}{\delta_R}+\sin^{-1}\frac{\delta_R}{R}\right)$$

for all $R>0$. For any fixed R we are free to choose $\delta_R>0$ arbitrarily small, and when $\delta_R' < \delta_R$,

$$||I(z(R,\delta_R'))|| \leq ||I(z(R,\delta_R))||.$$

Moreover

$$\sin^{-1}\frac{\delta_R}{R}\to 0$$

as $\delta_R\to 0$. Given any $\epsilon>0$, choose R sufficiently large that

$$\frac{34}{R}<\frac{\epsilon}{3},$$

then choose δ_R sufficiently small to ensure

$$\frac{2}{\pi}||I(z(R,\delta_R))||\sin^{-1}\frac{\delta_R}{R}<\frac{\epsilon}{3}.$$

This yields

$$||I(0)-g_T(0)|| \leq \frac{2\epsilon}{3}+\frac{2}{\pi}||I(z(R,\delta_R))||\frac{e^{-\delta_R T}R}{\delta_R}.$$

Fixing R, for all sufficiently large T we have

$$||I(0)-g_T(0)||<\epsilon.$$

We have proved

$$\forall\epsilon>0:\exists T_0: T\geq T_0 \implies ||I(0)-g_T(0)||<\epsilon.$$

Therefore

$$\lim_{T\to\infty} g_T(0)=I(0).$$

In other words,

$$\int_0^\infty \theta(e^t)e^{-t}-1\ dt = I(0).$$

$\square$