## Exercises for Math 411/511: Introduction to Complex Variables

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Textbook: Complex Analysis, by Elias M. Stein \& Rami Shakarachi

## The Field $\mathbb{C}$

1. Find all $z=a+b i$ satisfying $z^{2}-(3-i) z+(4-3 i)=0$.
2. Given $n \in \mathbb{N}$, simplify $\cos \theta+\cos 3 \theta+\cdots+\cos (2 n+1) \theta$.
3. Page 24, Problem 1.

## Sequences in $\mathbb{C}$

1. Prove that $\lim _{n \rightarrow \infty} \frac{n+i}{n-i}=1$ using an $\epsilon-N$ argument.

## The Sum, Product, and Quotient Rules

1. Prove, by induction on $n$, that if $f(z)=a_{0}+a_{1} z+\cdots+a_{n} z^{n}$ is a polynomial with complex coefficients and $\lim _{n \rightarrow \infty} z_{n}=c$ then $\lim _{n \rightarrow \infty} f\left(z_{n}\right)=f(c)$.

## A Brief Review of the Topology of $\mathbb{R}$

1. Let $a_{1}=2$, and for each $n \geq 1$ let $a_{n+1}=\frac{a_{n}}{2}+\frac{1}{a_{n}}$. (a) Prove that $\sqrt{2}<a_{n} \leq 2$ for all $n$. (b) Prove that $a_{n}>a_{n+1}$ for all $n$. (c) Prove that $\lim _{n \rightarrow \infty} a_{n}=\sqrt{2}$.
2. For each $n \in \mathbb{N}$ let $a_{n}=\cos (n)$, where $n$ has units of radians. Does the sequence $\left(a_{n}\right)$ have an increasing subsequence or a decreasing subsequence?

## Real and Complex Cauchy Sequences

1. For each $n \in \mathbb{N}$ let $a_{n}=\frac{(1+i)^{n}}{2^{n}}$. Let $\epsilon>0$ be given. Find $N$ so that $n>m \geq N \Longrightarrow\left\|a_{n}-a_{m}\right\|<\epsilon$.
Topology of $\mathbb{C}$
2. Prove that $[0,1]$ is closed when regarded as a subset of $\mathbb{C}$.
3. Prove that $(0,1)$ is not closed when regarded as a subset of $\mathbb{C}$.
4. Let $S=\{z \in \mathbb{C}: 1<\|z\|<2\}$. Prove that $S$ is an open subset of $\mathbb{C}$.

Compact Subsets of $\mathbb{C}$
1 Let $S=\{z \in \mathbb{C}: 2 \leq\|z\| \leq 3\}$. Prove that $S$ is a compact subset of $\mathbb{C}$.
2. Let $S=\{a+b i \in \mathbb{C}: 0 \leq a \leq 1,0 \leq b \leq 2\}$. Find the diameter of $S$.
3. Let $S$ be the set in Problem 2. Let $T=\{(1+i) z: z \in S\}$. Find the diameter of $T$.

## Complex Functions and Continuity

1. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z)=z^{2}$. Prove, using an $\epsilon-\delta$ argument, that $f$ is continuous at $1+i$. In other words, given $\epsilon>0$, find a formula for $\delta>0$ such that $\|z-1-i\|<\delta$ implies $\left\|z^{2}-(1+i)^{2}\right\|<\epsilon$.

## Holomorphic Complex Functions

1. Let $f: \mathbb{C}-\{0\} \rightarrow \mathbb{C}$ be defined by $f(z)=\frac{1}{z^{2}}$. Let $z_{0} \neq 0$ be given. Prove, using the definition of complex derivative, that $f^{\prime}\left(z_{0}\right)=\frac{-2}{z_{0}^{3}}$.
The Sum, Product, and Chain Rule for Complex Differentiation
2. Let $f: \mathbb{C}-\{i\} \rightarrow \mathbb{C}$ be defined by $f(z)=\left(\frac{z+i}{z-i}\right)^{3}$. Find $f^{\prime}(2 i)$.

## Some Real Analysis

1. Let $f:[-1,1] \rightarrow \mathbb{R}$ be defined by $f(x)=\frac{x}{x^{2}+1}$. (a) Graph the function. (b) Prove that $f$ is injective on $[-1,1]$. (c) Prove that $f([-1,1])=\left[-\frac{1}{2}, \frac{1}{2}\right]$.
(d) Find a formula for the function $g:\left[-\frac{1}{2}, \frac{1}{2}\right] \rightarrow[-1,1]$ that is the inverse of the function $f:[-1,1] \rightarrow\left[-\frac{1}{2}, \frac{1}{2}\right]$ defined by $f(x)=\frac{x}{x^{2}+1}$. Check that your answer is correct by verifying that $f(g(a))=g(f(a))=a$ for $a=\frac{1}{4}$ and $a=0$.

## Complex Extreme Value Theorem

1. Let $S=\{z \in \mathbb{C}: 2 \leq\|z\| \leq 3\}$. Define $f: S \rightarrow \mathbb{C}$ by $f(z)=\frac{1}{z-i}$. Find the minimum and maximum values of $\|f(z)\|$ on $S$.

## The Cauchy-Riemann Equations

1. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z)=z^{3}$. Verify that $f$ satisfies the Cauchy-Riemann equations at all $a+b i \in \mathbb{C}$.
2. Assume that the functions $u(x, y)$ and $v(x, y)$ are defined on the set

$$
B_{\epsilon}(a, b)=\left\{(x, y):(x-a)^{2}+(y-b)^{2}<\epsilon^{2}\right\}
$$

and that the partial derivatives $u_{x}, u_{y}, v_{x}, v_{y}$ exist at all $(x, y) \in B_{\epsilon}(a, b)$. Let $U(r, \theta)=u(r \cos \theta, r \sin \theta)$ and $V(r, \theta)=v(r \cos \theta, r \sin \theta)$ be defined for all $r$ and $\theta$ satisfying $(r \cos \theta, r \sin \theta) \in B_{\epsilon}(a, b)$. Prove that if the equations

$$
\frac{\partial U}{\partial r}=\frac{1}{r} \frac{\partial V}{\partial \theta}
$$

and

$$
\frac{1}{r} \frac{\partial U}{\partial \theta}=-\frac{\partial V}{\partial r}
$$

are satisfied at the point $(r, \theta)$ then $u(x, y)$ and $v(x, y)$ satisfy the CauchyRiemann equations at the point $(r \cos \theta, r \sin \theta)$.
3. Problem 9, page 27.
4. Problem 12, page 27.
5. Problem 13, page 28.

## The Complex Exponential Function

1. Prove that the function $e^{z}$ satisfies the Cauchy-Riemann equations at all $z \mathbb{C}$.
2. Find all solutions to $e^{z}=1+i$.

## Complex Trigonometric Functions

1. Verify the statements in the notes.
2. Find all solutions to $\cos z=\sin z$.

## The Complex Logarithm

1. Find all solutions to $\log (z)=1-i$.
2. Find all solutions to $\log _{\frac{\pi}{3}}(z)=1-i$.

## Exponentiation

1. Compute $2^{i}$.
2. Find all solutions to $z^{i}=1+i$ in the set $\mathbb{C}-\{x+i y: x \leq 0\}$.

## Series of Complex Numbers

1. Prove that the series $\sum_{n=1}^{\infty} \frac{(1+2 i)^{n}}{n 3^{n}}$ converges.
2. Prove that the series $\sum_{n=1}^{\infty} \frac{i^{n}}{n}$ converges.
3. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2^{n i}}$ diverges.
4. Prove that the series $\sum_{n=1}^{\infty} \frac{\left(1+2^{n i}\right)^{n}}{3^{n}}$ converges.

## Functions Defined by Power Series

1. Problem 16, page 28, using the root or ratio tests as needed.
2. Problem 19 ab, page 29.

## Functions Defined by Power Series are Infinitely Differentiable

1. Problem 20, page 29. First find the geometric series representation for $\frac{1}{1-z}$, then see what you obtain when you differentiate $m-1$ times. For the second part of the question, prove that $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=1$ where $b_{n}=\frac{1}{(m-1)!} n^{m-1}$.

## Complex Line Integrals

1. Problem 25ab, page 30 .
2. Problem 26, page 31. Use the Cauchy-Riemann equations to show that if $f^{\prime}(z)=0$ for all $z \in B_{r}\left(z_{0}\right)$ then $f(z)$ is constant on $z \in B_{r}\left(z_{0}\right)$. Use this to show that if $F^{\prime}(z)=f(z)$ and $G^{\prime}(z)=f(z)$ on $B_{r}\left(z_{0}\right)$ then $F(z)=G(z)+c$ for some constant $c$ on $B_{r}\left(z_{0}\right)$.
3. Problem 24, page 31. Prove that $\alpha:[0,1] \rightarrow \mathbb{C}$ is given by $\alpha(t)=$ $x(t)+i y(t)$ and if $\beta:[0,1] \rightarrow \mathbb{C}$ is defined by $\beta(t)=\alpha(1-t)$ then

$$
\int_{\beta} f(z) d z=-\int_{\alpha} f(z) d z .
$$

## Equivalent Paths

1. Let $\alpha:[0,2 \pi] \rightarrow \mathbb{C}$ be defined by $\alpha(\theta)=e^{i \theta}$ and let $\beta:[0, \sqrt{2 \pi}] \rightarrow \mathbb{C}$ be defined by $\beta(\theta)=e^{i \theta^{2}}$. Prove that $\alpha$ and $\beta$ are equivalent paths, then show by a direct computation that

$$
\int_{\alpha} \frac{1}{z} d z=\int_{\beta} \frac{1}{z} d z
$$

## Complex Line Integrals over Piecewise Smooth Paths

1. Let $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}:[-1,1] \rightarrow \mathbb{C}$ be defined by

$$
\alpha_{1}(t)=1+t i, \alpha_{2}(t)=-t+i, \alpha_{3}(t)=-1-t i, \alpha_{4}(t)=t-i
$$

(a) Sketch the piecewise smooth path $\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}$.
(b) Prove by a direct calculation that $\int_{\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}} z^{2} d z=0$.

## Change of Variables in a Line Integral

1. Find a formula for $\int_{\gamma} f(a z+b) d z$ where $a, b \in \mathbb{C}$ and $a \neq 0$.
2. Problem 25 c, page 30 . Use a partial-fraction decomposition and evaluate each piece separately.
3. (a) Let $z \neq 1$ be a complex number and let $R \geq 1$ be a real number. Prove

$$
\int_{1}^{R} \frac{1}{t^{z}} d t=\int_{0}^{\ln R} e^{u(1-z)} d u
$$

using the change of variables $u=\ln t$.
(b) Prove that

$$
\int_{0}^{\ln R} e^{u(1-z)} d u=\frac{1}{1-z} \int_{\gamma} e^{w} d w
$$

where $\gamma:[0, \ln R] \rightarrow \mathbb{C}$ is the path defined by $\gamma(u)=u(1-z)$.
(c) Use (a) and (b) to prove

$$
\int_{1}^{R} \frac{1}{t^{z}} d t=\frac{R^{1-z}-1}{1-z}
$$

(d) Assume $z=a+b i$ where $a>1$. Prove

$$
\int_{1}^{\infty} \frac{1}{t^{z}} d t=\frac{1}{z-1}
$$

## The $M-L$ Inequality

1. Let $f(z)=a_{0}+a_{1} z+\cdots a_{n}^{n}$ be a polynomial of degree $n$ (i.e. $a_{n} \neq 0$ ).
(a) Prove that

$$
\|f(z)\| \geq\left\|a_{n} z^{n}\right\|-\left(\left\|a_{0}\right\|+\left\|a_{1} z\right\|+\cdots+\left\|a_{n-1} z^{n-1}\right\|\right)
$$

for all $z$.
(b) Prove that there exists $R>0$ such that

$$
\|z\| \geq R \Longrightarrow\left\|a_{0}\right\|+\left\|a_{1} z\right\|+\cdots+\left\|a_{n-1} z^{n-1}\right\| \leq \frac{1}{2}\left\|a_{n} z^{n}\right\|
$$

(c) Using (a) and (b), prove that there exists $R>0$ such that

$$
\|z\| \geq R \Longrightarrow\|f(z)\| \geq \frac{1}{2}\left\|a_{n}\right\| R^{n}
$$

2. Let $f(z)=a_{0}+a_{1} z+\cdots+a_{n} z^{n}$ and $g(z)=b_{0}+b_{1} z+\cdots+b_{m} z^{m}$ be polynomials of degrees $n$ and $m$, respectively. Using Problem 1, part (c) prove that there exists $R>0$ such that

$$
\|z\| \geq R \Longrightarrow\left\|\frac{f(z)}{g(z)}\right\| \leq \frac{2\left(\left\|a_{0}\right\|+\left\|a_{1}\right\|+\cdots+\left\|a_{n}\right\|\right)}{\left\|b_{m}\right\|} R^{n-m}
$$

3. Using the $M-L$ inequality, prove that for sufficiently large $R$,

$$
\int_{C_{R}(0)} \frac{z^{3}-4 z+1}{\left(z^{2}+5\right)\left(z^{3}-3\right)} d z \leq \frac{24 \pi}{R}
$$

where $C_{R}(0)$ denotes the path $\gamma:[0,2 \pi] \rightarrow \mathbb{C}$ defined by $\gamma(t)=R e^{i t}$.
4. Let $z=a+b i$ be a complex number, and let $n$ be a positive integer. Prove that when $n \leq x \leq n+1$,

$$
\left\|\frac{1}{n^{z}}-\frac{1}{x^{z}}\right\| \leq \frac{\|z\|}{n^{a+1}}
$$

Method: Let $\gamma:[n, t]$ be defined by $\gamma(x)=\frac{1}{x^{z}}$. Use the $M-L$ inequality combined with the equation

$$
\frac{1}{n^{z}}-\frac{1}{x^{z}}=\gamma(x)-\gamma(n)=\int_{n}^{x} \gamma^{\prime}(t) d t
$$

## Complex Line Integrals over Straight Line Paths

1. Verify the claims made in the proof of the lemma in this section of the notes, i.e. that $\alpha$ is equivalent to $\gamma_{z_{1}, z_{2}}$ and $\beta$ is equivalent to $\gamma_{z_{2}, z_{3}}$.

## Goursat's Theorem

1. Prove that $B_{r}(a)$ is a convex subset of $\mathbb{C}$.
2. Let $z_{1}=0+0 i, z_{2}=1+0 i$, and $z_{3}=1+1 i$.
(a) Let $f(x+i y)=x^{2}-y^{2}+3 x y i$. Prove by a direct calculation that

$$
\int_{\gamma_{z_{1}, z_{3}}} f(z) d z \neq \int_{\gamma_{z_{1}, z_{2}}} f(z) d z+\int_{\gamma_{z_{2}, z_{3}}} f(z) d z
$$

If $f$ holomophic on $\mathbb{C}$ ?
2. Let $z_{1}=0+0 i, z_{2}=1+0 i$, and $z_{3}=1+1 i$.
(a) Let $f(x+i y)=x^{2}-y^{2}+2 x y i$. Prove by a direct calculation that

$$
\int_{\gamma_{z_{1}, z_{3}}} f(z) d z=\int_{\gamma_{z_{1}, z_{2}}} f(z) d z+\int_{\gamma_{z_{2}, z_{3}}} f(z) d z
$$

If $f$ holomophic on $\mathbb{C}$ ?

## Antiderivative Construction in an Open Convex Set

1. Let $S=\{x+i y \in \mathbb{C}: x>1\}$. Then $S$ is a convex open set, and the holomorphic function $f: S \rightarrow \mathbb{C}$ defined by $f(z)=\frac{1}{z}$ has antiderivative $F: S \rightarrow \mathbb{C}$ defined by

$$
F(z)=\int_{\gamma_{2, z}} \frac{1}{z} d z
$$

where $\gamma_{2, z}$ parameterizes the line between 2 and $z$. Now suppose $z=a+b i \in$ $S$. Then by Goursat's Theorem,

$$
F(z)=\int_{\gamma_{2,2+b i}} \frac{1}{z} d z+\int_{\gamma_{2+b i, a+b i}} \frac{1}{z} d z
$$

By evaluating the latter expression, derive a formula for $F(a+b i)$.

## Cauchy's Theorem in an Open Convex Set

1. Problem 2, page 64. Use the technique of Example 2, page 44, discussed in these notes, applied to the function $f(z)=\frac{e^{i z}-1}{z}$.
2. Problem 3, page 64. There is an alternative way to do this problem: let $\gamma_{R}:[0, R] \rightarrow \mathbb{C}$ be defined by $\gamma_{R}(t)=t$. Compute $\int_{\gamma_{R}} e^{(-a+b i) z} d z=c_{R}$ by using an appropriate antiderivative, argue that this implies

$$
\int_{0}^{R} e^{-a x} \cos (b x) d x+i \int_{0}^{R} e^{-a x} \sin (b x) d x=c_{R}
$$

then let $R \rightarrow \infty$.

## Cauchy's Integral Formula

1. Let $f: \mathbb{C}-\{0\}$ be defined by $f(z)=\frac{1}{z}$. Then $f$ meets the hypotheses of Cauchy's Theorem on $D_{1}(2)$. Verify that for all $z \in B_{1}(2)$,

$$
f(z)=\frac{1}{2 \pi i} \int_{C_{1}(2)} \frac{f(w)}{w-z} d w
$$

i.e.

$$
\frac{1}{z}=\frac{1}{2 \pi i} \int_{C_{1}(2)} \frac{1}{w(w-z)} d w
$$

by evaluating the integral using a partial fraction decomposition.

## Sequences of Functions

1. Let $\left(h_{n}\right)$ be an arbitrary sequence of non-zero complex numbers satisfying $\lim _{n \rightarrow \infty} h_{n}=0$. For each $n \in \mathbb{Z}$ define $f_{n}: \mathbb{C} \rightarrow \mathbb{C}$ by

$$
f_{n}(z)=\frac{\left(z+h_{n}\right)^{2}-z^{2}}{h_{n}}
$$

Define $f: \mathbb{C} \rightarrow \mathbb{C}$ by $f(z)=2 z$. Prove that $\left(f_{n}\right)$ converges to $f$ uniformly on $\mathbb{C}$.
2. Let $\left(h_{n}\right)$ be an arbitrary sequence of non-zero complex numbers satisfying $\lim _{n \rightarrow \infty} h_{n}=0$. For each $n \in \mathbb{Z}$ define $f_{n}: \mathbb{C}-\{0\} \rightarrow \mathbb{C}$ by

$$
f_{n}(z)=\frac{\frac{1}{z+h_{n}}-\frac{1}{z}}{h_{n}} .
$$

Define $f: \mathbb{C} \rightarrow \mathbb{C}$ by $f(z)=\frac{-1}{z^{2}}$. Prove that $\left(f_{n}\right)$ does not converge to $f$ uniformly on $\mathbb{C}-\{0\}$.
3. Fix $a+b i \in \mathbb{C}$ where $a>0$. For each $n \in \mathbb{N}$ define $f: B_{\frac{a}{2}}(a+b i) \rightarrow \mathbb{C}$ by

$$
f_{n}(z)=\int_{n}^{n+1} \frac{1}{n^{z}}-\frac{1}{x^{z}} d x .
$$

Using the Weierstrass $M$-Test, prove that $\sum_{n=1}^{\infty} f_{n}$ converges uniformly on $B_{\frac{a}{2}}(a+b i)$. Make use of the information in Problem 4 in the section on the $M-L$ inequality.
4. Fix $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Prove that

$$
\sum_{k=0}^{\infty}\left[\int_{n}^{n+1} \frac{(\ln n)^{k}-(\ln x)^{k}}{k!}(-z)^{k} d x\right]=\int_{n}^{n+1}\left[\sum_{k=0}^{\infty} \frac{(\ln n)^{k}-(\ln x)^{k}}{k!}(-z)^{k}\right] d x
$$

Method: For each $k \geq 0$ define $f_{k}:\{x+y i \in \mathbb{C}: x>0\} \rightarrow \mathbb{C}$ via

$$
f_{k}(w)=\frac{(\ln n)^{k}-(\log w)^{k}}{k!}(-z)^{k}
$$

Apply the Weierstrass $M$-test to $\left(f_{k}\right)$ on the set $S=[n, n+1]$, then justify exchanging the order of summation and integration in the equation

$$
\sum_{k=0}^{\infty} \int_{\gamma} f_{k}(w) d w=\int_{\gamma} \sum_{k=0}^{\infty} f_{k}(w) d w
$$

where $\gamma:[n, n+1] \rightarrow \mathbb{C}$ is defined by $\gamma(x)=x$.

## Power Series Expansion of Holomorphic Functions

1. Let $c$ be a fixed complex number. Define $f: B_{1}(0) \rightarrow \mathbb{C}$ by $f(z)=(1+z)^{c}$. Find a power series expansion for $f$.
2. Fix $t>0$. Find the power series expansion of the function $f: \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z)=\frac{1}{t^{z}}$.
3. Fix $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Prove that

$$
\sum_{k=0}^{\infty}\left[\int_{n}^{n+1} \frac{(\ln n)^{k}-(\ln x)^{k}}{k!}(-z)^{k} d x\right]=\int_{n}^{n+1} \frac{1}{n^{k}}-\frac{1}{z^{k}} d x
$$

4. Fix $a+b i \in \mathbb{C}$ with $a>0$. Fix $n \in \mathbb{N}$. For each $k \geq 0$ define $f_{k}$ : $B \frac{a}{2}(a+b i) \rightarrow \mathbb{C}$ by

$$
f_{k}(z)=\int_{n}^{n+1} \frac{(\ln n)^{k}-(\ln x)^{k}}{k!}(-z)^{k} d x
$$

Prove that $\left(\sum_{k=0}^{\infty} f_{k}\right)$ converges uniformly to the function $f: B_{\frac{a}{2}}(a+b i) \rightarrow \mathbb{C}$ defined by

$$
f(z)=\int_{n}^{n+1} \frac{1}{n^{z}}-\frac{1}{x^{z}} d x
$$

## Power Series Expansions of Products and Quotients

1. Let $f: B_{1}(0) \rightarrow \mathbb{C}$ be defined by $f(z)=\frac{e^{z}}{\cos \frac{\pi}{2} z}$, and suppose that the power series expansion for $f(z)$ is $\sum_{n=0}^{\infty} a_{n} z^{n}$. Compute $a_{0}$ through $a_{5}$.
Liouville's Theorem and The Fundamental Theorem of Algebra
2. Factor the following polynomials into linear factors of the form $z-(a+b i)$ where $a, b \in \mathbb{R}$ : (a) $z^{3}+z^{2}+z+1$, and (b) $z^{5}+z^{4}+z^{3}+z^{2}+z+1$.

## Laurent Series

1. Consider the function $f: \mathbb{C}-\{0, i,-i\}$ defined by

$$
f(z)=\frac{z+1}{z^{4}+z^{2}}
$$

Compute each of the coefficients of the Laurent expansion of $f(z)$ when expanded in powers of: (a) $z$, and (b) $z-i$.

## The Residue Theorem

1. Using the Residue Theorem, compute the following three integrals:
(a) $\int_{C_{\frac{1}{2}}(0)} \frac{z+1}{z^{4}+z^{2}} d z$.
(b) $\int_{C_{\frac{1}{2}}(1)} \frac{z+1}{z^{4}+z^{2}} d z$.
(c) $\int_{C_{\frac{1}{2}}(i)} \frac{z+1}{z^{4}+z^{2}} d z$.

## Computing Residues

1. Compute the residue of $f(z)=\frac{1}{z^{6}+3 z^{4}+3 z^{2}+1}$ at $z=i$.
2. Compute the residue of $f(z)=\frac{z-\pi}{1+\cos z}$ at $z=\pi$.

## Generalized Residue Theorem

1. Evaluate the following integrals:
(a) $\int_{C_{1}(0)} \frac{e^{k z}}{z^{n+1}} d z$.
(b) $\int_{C_{2}(0)} \frac{z^{3}}{z^{2}-2 z+2} d z$.
(c) $\int_{C_{3}(0)} \frac{e^{z}}{\pi i-2 z} d z$.
(d) $\int_{C_{4}(0)} \frac{\sin ^{2} z}{\left(z-\frac{\pi}{6}\right)^{2}\left(z+\frac{\pi}{6}\right)} d z$.

## Trigonometric Integrals

1. Evaluate $\int_{0}^{2 \pi} \frac{d \theta}{10+\cos \theta}$ by contour integration of an appropriate complex integrand.
2. Problem 8, page 104.
3. Problem 7, page 104.

## Improper Integrals

1. Explain why the improper integral $\int_{0}^{\infty} \frac{d x}{5+x^{4}}$ converges.
2. Explain why P.V. $\int_{-\infty}^{\infty} \frac{x+1}{5+x^{4}} d x=\int_{-\infty}^{\infty} \frac{x+1}{5+x^{4}} \mathrm{dx}$.
3. Explain why P.V. $\int_{-\infty}^{\infty} \frac{x^{9}}{1+x^{2}} d x \neq \int_{-\infty}^{\infty} \frac{x^{9}}{1+x^{2}} d x$.

## Improper Integrals and Semicircular Paths

I. Prove $\int_{0}^{\infty} \frac{1}{1+x^{4}} d x=\frac{\sqrt{2}}{4} \pi$.

Hint: $z^{4}+1=\left(z^{2}-i\right)\left(z^{2}-i\right)$. To factor $z^{2}-i$, use the fact that $i=e^{\frac{\pi}{2} i}$. To factor $z^{2}+i$, use the fact that $-i=e^{-\frac{\pi}{2} i}$.
II. Prove $\int_{0}^{\infty} \frac{\cos x}{\left(1+x^{2}\right)^{2}} d x=\frac{\pi}{2 e}$.
III. a Prove $\int_{0}^{\infty} \frac{\sin x}{x\left(1+x^{2}\right)} d x=\pi \frac{e-1}{2 e}$.

Hint: Integrate the function $f(z)=\frac{e^{i z}-1}{z\left(1+z^{2}\right)}$ over the indented contour. Show that the sum over the two paths along the $x$-axis is equal to $2 i \int_{r}^{R} \frac{\sin x}{x\left(1+x^{2}\right)} d x$. Use a differential approximation of $\frac{e^{i z}-1}{z}$ near the origin as we did for the evaluation of $\int_{0}^{\infty} \frac{1-\cos x}{x^{2}} d x$ to show that $\lim _{r \rightarrow 0} \int_{\beta_{r}} \frac{e^{i z}-1}{z\left(1+z^{2}\right)} d z=\lim _{r \rightarrow 0} \int_{\beta_{r}} \frac{i}{1+z^{2}} d z=$ 0 using the $M-L$ inequality.
III.b Problem 10, page 104.
IV. Prove $\int_{0}^{\infty} \frac{1}{1+x^{3}} d x=\frac{2 \sqrt{3}}{9} \pi$.

Hint: Integrate $f(z)=\frac{1}{1+z^{3}}$ over the contour $\gamma=\alpha+\beta-\delta$, where $\alpha(t)=t$ on $[0, R], \beta(t)=R e^{i t}$ on $\left[0, \frac{2 \pi i}{3}\right]$, and $\delta(t)=t e^{\frac{2 \pi i}{3}}$ on $[0, R]$, then let $R \rightarrow \infty$.

## Improper Integrals and Rectangular Paths

1. Problem 3, page 103.

## Rectangular Paths of Fixed Width

1. Let $0<a<1$. By considering the integral of $f(z)=\frac{z}{a-e^{-i z}}$ around the rectangle with sides defined by $x=-\pi, x=\pi, y=0, y=R>0$, show that

$$
\int_{0}^{\pi} \frac{t \sin t}{1-2 a \cos t+a^{2}} d t=\frac{\pi}{a} \ln (1+a) .
$$

Hint: Show that $f(z)$ is holomorphic above and on the $x$-axis, i.e. that $a-e^{-i z} \neq 0$, therefore the integral around the contour is 0 . Show that the integral along the $y=R$ side of the integral tends to 0 as $R \rightarrow \infty$ using the $M-L$ inequality. Show that the real part of the sum of the integrals over the vertical sides of the rectangle is equal to zero. You can evaluate the imaginary part of the sum of the integrals over the vertical sides using the substitution $u=a+e^{t}$ and partial fraction decomposition.

## Some Infinite Series Evaluations

1. Evaluate $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{8}}$.
2. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k^{8}}$.
3. Using the function $f(z)=\frac{1}{(2 z+1)^{5} \sin \pi z}$, prove $\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k+1)^{5}}=\frac{5}{1536} \pi^{5}$.

## Analytic Continuation of Holomorphic Functions

1. Let $f: \mathbb{C}-\{3\} \rightarrow \mathbb{C}$ be defined by

$$
f(z)=\frac{e^{z}-e^{3}(z-2)}{(z-3)^{2}}
$$

Find an analytic continuation of $f$ to $\mathbb{C}$.

## The Riemann Zeta Function

1. Evaluate $\zeta(2)$.
2. Evaluate $\prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{p_{n}^{3}}}$, where $\left(p_{n}\right)$ is the sequence of prime numbers.
3. Prove that

$$
\zeta_{1}\left(\frac{1}{2}\right)=\lim _{n \rightarrow \infty}\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2 \sqrt{n+1}\right)
$$

Plot the first 100 terms of the sequence.
4. Prove that

$$
\operatorname{re} \zeta_{1}(1+i)=\lim _{n \rightarrow \infty}\left(\frac{\cos \ln 1}{1}+\frac{\cos \ln 2}{2}+\cdots+\frac{\cos \ln n}{n}-\sin \ln (n+1)\right) .
$$

Plot the first 100 terms of the sequence.

## The Prime Number Theorem

1. Prove that $F(1)=\lim _{n \rightarrow \infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln (n+1)\right)$. Plot the first 100 terms of the sequence.
