Final Exam Solutions Math 121 Spring 2015 Name:

Work must be shown to receive credit.

1. Let $f(x) = 5x^7 - 63x^5$. (a) Make a sign chart for $f(x)$. (b) Find all local maximum and minimum values of $f(x)$. (c) Find all points of inflection on the curve $y = f(x)$.

Solution: (a) $f'(x) = 35x^6 - 315x^4 = 35x^4\left(x^2 - 9\right)$ has roots $0, \pm 3$. $f''(x) = 210x^5 - 1260x^3 = 210x^3\left(x^2 - 6\right)$ has roots $0, \pm\sqrt{6}$. Sign chart is

x	$\cdots$	-3	$\cdots$	$-\sqrt{6}$	$\cdots$	0	$\cdots$	$\sqrt{6}$	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$	$+$

Local max value $f(-3)$, local min value $f(3)$, points of inflection $(-\sqrt{6}, f(-\sqrt{6})), (0, f(0)), (\sqrt{6}, f(\sqrt{6}))$.

2. If 900 in^2 of material is available to make a box with square base and an open top, find the largest possible volume of the box. You must use calculus to receive credit for this problem.

Solution: The dimensions of the box are x by x by h, h to be determined. The area of the base is x^2 and the area of each of the other four sides is xh, therefore $x^2 + 4xh = 900$, therefore $h = \frac{900-x^2}{4x}$. Hence the volume of the box is $x^2h = x^2\left(\frac{900-x^2}{4x}\right) = \frac{1}{4}(900x - x^3)$. To ensure a positive height we need $0 < x < 30$. There is no harm in allowing x to be 0 or 30, because this will not affect maximum volume. In order to find the maximum value of $f(x) = \frac{1}{4}(900x - x^3)$ on $[0, 30]$ we need only check $f(0)$, $f(30)$, and $f(c)$ where $f'(c) = 0$. Given $f'(x) = \frac{1}{4}(900 - 3x^2)$, we have $c = \sqrt{300}$. We can rule out $f(0) = f(30) = 0$, therefore maximum volume is $f(\sqrt{30}) = 1500\sqrt{3} = 2598.08$ in^3.

3. Let $f(x) = x^4 + x^2 - x$. Find the minimum value of $f(x)$ on the interval $[0, 1]$. Use Newton's Method with $x_0 = 0.8$ to find the critical value of the function, accurate to 2 decimal places.

Solution: Checking the boundary of the interval, $f(0) = 0$ and $f(1) = 1$. Critical value occurs where $f'(x) = 0$, i.e.

$$4x^3 + 2x - 1 = 0.$$

We solve this equation using Newton's Method applied to $F(x) = 4x^3 + 2x - 1$. Since $F'(x) = 12x^2 + 2$, Newton's Method yields

$$x_{n+1} = x_n - \frac{4x_n^3 + 2x_n - 1}{12x_n^2 + 2}.$$

Using $x_0 = 0.8$ we obtain $x_1 = 0.526446$, $x_2 = 0.406932$, $x_3 = 0.386013$, $x_4 = 0.385459$. The approximate critical value is 0.39. $f(0.39) = -0.214766$, so the minimum value of the function is approximately -0.21.

4. A car with velocity $v(t) = k - (k/3)t$ feet per second at time t seconds travels 200 feet from time $t = 0$ to time $t = 3$. Using the relationship between distance and area, find k.

Solution: By our association with area under the velocity curve and distance travelled, 200 must equal the area under $v(t) = k - (k/3)t$ for $0 \leq t \leq 3$. The latter is a triangle of height k, length 3. Therefore $200 = (1/2)(k)(3) = 1.5k$, $k = \frac{200}{1.5} = 133.333$ feet per second.

5. Approximate the definite integral $\int_0^2 2^x \, dx$ using a Riemann Sum with $N = 4$ equally spaced subintervals and the midpoint

rule to generate x_1^* through x_4^*. Round your answer to two decimal places.

Solution: The partition points are $x_i = 0, 0.5, 1.0, 1.5, 2.0$ and the midpoint rule yields $x_i^* = 0.25, 0.75, 1.25, 1.75$. The resulting Riemann Sum is

$$\left(2^{0.25} + 2^{0.75} + 2^{1.25} + 2^{1.75}\right) 0.5 \approx 4.31.$$

6. Find the exact value of $\int_1^2 \frac{x+x^2}{x^3}\, dx$.

Solution: The integrand is $f(x) = \frac{x+x^2}{x^3} = x^{-2} + x^{-1}$. An antiderivative is $F(x) = \frac{x^{-1}}{-1} + \ln x = \ln x - \frac{1}{x}$. The definite integral is $F(2) - F(1) = (\ln 2 - \frac{1}{2}) - (\ln 1 - \frac{1}{1}) = \ln 2 + \frac{1}{2}$.

7. Oil leaks from a tank at a rate of $1 + \sin t$ gallons per hour at time t hours. Find the total amount of oil that has leaked from time $t = \frac{\pi}{2}$ to time $t = \pi$ hours.

Solution: Let $G(t) =$ number of gallons of leaked oil at time t. We are given $G'(t) = 1 + \sin t$, hence the net increase is

$$G(\pi) - G(\frac{\pi}{2}) = \int_{\frac{\pi}{2}}^{\pi} 1 + \sin t\ dt = t - \cos t|_{\frac{\pi}{2}}^{\pi} =$$

$$(\pi - \cos(\pi)) - (\frac{\pi}{2} - \cos(\frac{\pi}{2})) = 1 + \frac{\pi}{2} \text{ gallons.}$$

8. Evaluate the definite integral $\int_0^1 \frac{x^2}{(1+x^3)^2}\, dx$ using u-substitution.

Solution: $u = 1 + x^3$, $du = 3x^2\, dx$, $\frac{1}{3}du = x^2\, dx$, $u(0) = 1$, $u(1) = 2$,

$$\frac{x^2}{(1+x^3)^2}\, dx = \frac{1}{(1+x^3)^2} \cdot x^2\, dx = \frac{1}{u^2} \cdot \frac{1}{3} du,$$

$$\int_0^1 \frac{x^2}{(1+x^3)^2}\ dx = \int_1^2 \frac{1}{3}u^{-2}\ du = \frac{1}{3}\frac{u^{-1}}{-1}\Bigg|_1^2 =$$

$$(\frac{1}{3})(\frac{-1}{2}) - (\frac{1}{3})(\frac{-1}{1}) = \frac{1}{6}.$$