MATH 542 NUMBER THEORY
Problems to Think About #6
CH. 6, #1-3

Russell Jahn

(1)

$b \nmid a \Rightarrow n \geq 1$.

$\frac{a}{b} = [a_0, a_1, ..., a_n] = \frac{p_n}{q_n}$ where $(p_n, q_n) = 1$.

Let $d = (a, b)$.

$a = d(\frac{a}{d})$, $b = d(\frac{b}{d})$.

$\exists u, v \in \mathbb{Z} \ni d = au + bv \Rightarrow 1 = \frac{a}{d}u + \frac{b}{d}v \Rightarrow (\frac{a}{d}, \frac{b}{d}) = 1$

$\Rightarrow \frac{a}{d} = p_n$, $\frac{b}{d} = q_n$.

We also have $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$.

$\Rightarrow \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n-1}} \Rightarrow \left| \frac{p_{n-1}}{q_{n-1}} - \frac{a}{b} \right| = \frac{1}{q_n q_{n-1}}$

$\Rightarrow |aq_{n-1} - bp_{n-1}| = \frac{bq_{n-1}}{q_n q_{n-1}} = \frac{b}{q_n} = d.$ ✓

(2)

We have $p_{k-1}q_k - p_k q_{k-1} = (-1)^k \Rightarrow \frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}$.

$$\Rightarrow \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}} = -\left(\left(\frac{p_0}{q_0} - \frac{p_1}{q_1} \right) + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2} \right) + ... + \left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right) \right) = -\frac{p_0}{q_0} + \frac{p_n}{q_n} = -a_0 + \frac{p_n}{q_n}$$

$$\Rightarrow \frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}}.$$

Moreover, $\theta = \lim_{n \to \infty} \frac{p_n}{q_n} = \lim_{n \to \infty} \left(a_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}} \right) = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k q_{k-1}}$. ✓

(3)

(a)

We prove by induction on n.

Base case: $n = 1$

$$\begin{bmatrix} p_1 & p_0 \\ q_1 & q_0 \end{bmatrix} = \begin{bmatrix} a_0a_1 + b_1 & a_0 \\ a_1 & 1 \end{bmatrix} = \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$$

Inductive hypothesis: Assume $\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & b_n \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$ for some $n \geq 1$.

$$\Rightarrow \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & b_n \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} p_{n-1} & p_n - a_np_{n-1} \\ q_{n-1} & q_n - a_nq_{n-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & b_{n+1} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_{n-1} & p_n - a_np_{n-1} \\ q_{n-1} & q_n - a_nq_{n-1} \end{bmatrix}\begin{bmatrix} a_n & b_{n+1} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_na_{n+1} + p_{n-1}b_{n+1} & p_n \\ q_na_{n+1} + q_{n-1}b_{n+1} & q_n \end{bmatrix} = \begin{bmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{bmatrix}.$$

$\therefore$ By induction, true for all $n \geq 1$. ✓

(b)

We prove for integers $a_i, b_i > 0$ for $i \geq 1$, that $[a_0, a_1, ..., a_n; b_1, b_2, ..., b_n] = \frac{p_n}{q_n}$ $\forall n \geq 0$
where $p_0 = a_0, p_1 = a_0a_1 + b_1, q_0 = 1, q_1 = a_1$ and
$p_n = a_np_{n-1} + b_np_{n-2}, \quad q_n = a_nq_{n-1} + b_nq_{n-2}$ for $n \geq 2$.

We induct on n, the number of elements b_i. Important note: n is not necessarily the subscript of the last a_i or b_i nor are a_0, b_0 necessarily the first a_i, b_i.

Base cases: n=0,1

$n = 0, \quad [a_0; -] = a_0 = \frac{p_0}{q_0}$.

$n = 1, \quad [a_0, a_1; b_1] = a_0 + \frac{b_1}{[a_1; -]} = a_0 + \frac{b_1}{a_1} = \frac{a_0a_1 + b_1}{a_1} = \frac{p_1}{q_1}$.

Inductive hypothesis: Assume, for any set of $a_i, b_i \in \mathbb{Z}$ such that $a_i, b_i > 0$ for $i \geq 1$,
that $[a_0, a_1, ..., a_n; b_1, b_2, ..., b_n] = \frac{p_n}{q_n}$
where $p_0 = a_0, p_1 = a_0a_1 + b_1, q_0 = 1, q_1 = a_1$ and
$p_n = a_np_{n-1} + b_np_{n-2}, \quad q_n = a_nq_{n-1} + b_nq_{n-2}$ for $n \geq 2$

is true for some $n \geq 1$, regardless of the subscripts used.

Consider $[a_0, ...a_{n+1}; b_1, ..., b_{n+1}] = a_0 + \frac{b_1}{[a_1, ..., a_{n+1}; b_2, ..., b_{n+1}]} = a_0 + \frac{b_1}{\frac{p'_n}{q'_n}} = \frac{a_0p'_n + b_1q'_n}{p'_n}$,

where $\frac{p'_n}{q'_n} = [a_1, ..., a_{n+1}; b_2, ..., b_{n+1}]$ by inductive hypothesis.

By part (a), $\begin{bmatrix} p'_n & p'_{n-1} \\ q'_n & q'_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & b_2 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & b_n \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_n & b_{n+1} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}$ and,

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} a_1 & b_2 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & b_n \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} p'_n & p'_{n-1} \\ q'_n & q'_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} a_n & b_{n+1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \tfrac{1}{b_1} \begin{bmatrix} b_1 q_n a_{n+1} + b_1 q_{n-1} b_{n+1} & b_1 q_n \\ p_n a_{n+1} - a_0 q_n a_{n+1} + p_{n-1} b_{n+1} - a_0 q_{n-1} b_{n+1} & p_n - a_0 q_n \end{bmatrix}.$$

$\Rightarrow [a_0, ...a_{n+1}; b_1, ..., b_{n+1}] = \frac{a_0 p'_n + b_1 q'_n}{p'_n} = \frac{a_{n+1} p_n + b_{n+1} p_{n-1}}{a_{n+1} q_n + b_{n+1} q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}.$

$\therefore$ By induction, the statement is proved. ✓

(c)

From part (a), $\begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} = \begin{vmatrix} a_0 & b_1 \\ 1 & 0 \end{vmatrix} \cdots \begin{vmatrix} a_{n-1} & b_n \\ 1 & 0 \end{vmatrix} \begin{vmatrix} a_n & 1 \\ 1 & 0 \end{vmatrix}$

$\Rightarrow p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \prod_{i=1}^{n} b_i \Rightarrow \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \frac{\prod_{i=1}^{n} b_i}{q_n q_{n-1}}$ for $n \geq 1$.

So, $a_0 + \sum_{k=1}^{n} (-1)^{k-1} \frac{\prod_{i=1}^{k} b_i}{q_k q_{k-1}} = a_0 + \left(\left(\frac{p_1}{q_1} - \frac{p_0}{q_0} \right) + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1} \right) + ... + \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right) \right) = a_0 - \frac{p_0}{q_0} + \frac{p_n}{q_n} = \frac{p_n}{q_n}.$ ✓

(d)

(1)

For $k \geq 1$, we have

$\frac{c_{k+1} - c_k}{c_k} = \frac{\frac{q_{k+1} q_k}{\prod_{i=1}^{k+1} b_i} - \frac{q_k q_{k-1}}{\prod_{i=1}^{k} b_i}}{\frac{q_k q_{k-1}}{\prod_{i=1}^{k} b_i}} = \frac{\frac{q_{k+1} q_k}{b_{k+1}} - q_k q_{k-1}}{q_k q_{k-1}} = \frac{q_{k+1} - b_{k+1} q_{k-1}}{b_{k+1} q_{k-1}}$

$= \frac{a_{k+1} q_k + b_{k+1} q_{k-1} - b_{k+1} q_{k-1}}{b_{k+1} q_{k-1}} = \frac{a_{k+1}}{b_{k+1}} \frac{q_k}{q_{k-1}}.$ ✓

(2)

For $k \geq 2$, we have

$q_k = a_k q_{k-1} + b_k q_{k-2} \Rightarrow \frac{q_k}{q_{k-1}} = a_k + b_k \frac{q_{k-2}}{q_{k-1}}$

$\Rightarrow \frac{b_{k+1}}{a_{k+1}} \frac{c_{k+1} - c_k}{c_k} = a_k + b_k \frac{a_k}{b_k} \frac{c_{k-1}}{c_k - c_{k-1}} = a_k \left(1 + \frac{c_{k-1}}{c_k - c_{k-1}} \right) = a_k \left(\frac{c_k}{c_k - c_{k-1}} \right)$

$\Rightarrow \frac{a_{k+1} a_k}{b_{k+1}} = \frac{(c_{k+1} - c_k)(c_k - c_{k-1})}{c_k^2}.$ ✓

(e)

First, we want to show $q_k = x_1 x_2 \cdots x_k \ \forall k \geq 1$.

We induct on k.

Base cases: $k = 1, 2$

$q_1 = a_1 = x_1, \quad q_2 = a_2 q_1 + b_2 q_0 = (x_2 - x_1) x_1 + x_1^2 1 = x_1 x_2.$

Inductive hypothesis: Assume $q_m = x_1 x_1 \cdots x_m \ \forall m \leq$ some $k \geq 2$.

$q_{k+1} = a_{k+1} q_k + b_{k+1} q_{k-1} = (x_{k+1} - x_k) x_1 \cdots x_k + x_k^2 x_1 \cdots x_{k-1} = x_1 \cdots x_{k+1}.$

$\therefore$ By strong induction, $q_k = x_1 x_2 \cdots x_k \ \forall k \geq 1$.

So, $c_k = \frac{q_k q_{k-1}}{\prod_{i=1}^{k} b_i} = \frac{(x_1 \cdots x_k)(x_1 \cdots x_{k-1})}{1 x_1^2 \cdots x_{k-1}^2} = x_k.$

$\therefore \frac{p_n}{q_n} = a_0 + \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{c_k} = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{x_k}$

Since $\left\{\frac{1}{x_k}\right\}_{k=1}^{\infty}$ is a positive, decreasing sequence that approaches 0, $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{x_k}$ is a convergent series by the alternating series test.

$\therefore a_0 + \frac{b_1}{a_1+\frac{b_2}{a_2+\frac{b_3}{a_3+\cdots}}} = \lim\limits_{n\to\infty} [a_0, a_1, ..., a_n; b_1, ..., b_n] = \lim\limits_{n\to\infty} \frac{p_n}{q_n} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{x_k} = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{x_k}$. ✓

(f)

We have the well known identity: $\ln(2) = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$.

In part (e), letting $x_k = k \Rightarrow a_k = 1$, $b_k = (k-1)^2$ for $k \geq 2$ and $a_0 = 0, a_1 = 1, b_1 = 1$

$\Rightarrow \ln(2) = \frac{1}{1+\frac{1}{1+\frac{4}{1+\frac{9}{1+\frac{16}{1+\cdots}}}}}$. ✓