MATH 542 NUMBER THEORY
Problems to Think About #2
CH. 2, #1-3

Russell Jahn

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(1)

f, g multiplicative

$h(n) = \sum_{d|P(n)} f(d)g(\frac{n}{d})$

$P(n) = r_1 r_2 \cdots r_v$ distinct primes that $|n$

$P(1) = 1$

Let $(a, b) = 1$

Let $a = p_1^{\alpha_1} ... p_k^{\alpha_k}$ $\quad p_i$ distinct, $\alpha_i > 0$

$b = q_1^{\beta_1} ... q_l^{\beta_l}$ $\quad q_j$ distinct, $\beta_j > 0$

$p_i \neq q_j \quad \forall i, j$

Let $\bar{a} = p_1 ... p_k, \bar{b} = q_1 ... q_l \quad \therefore (\bar{a}, \bar{b}) = 1$

First, we show $d|\bar{a}\bar{b} \iff d = d^{'} d^{''}$ where $d^{'}|\bar{a}, d^{''}|\bar{b}$

$(\Rightarrow) d|\bar{a}\bar{b} \Rightarrow d = p_{i_1} ... p_{i_m} \cdot q_{j_1} ... q_{j_n}$

where $i_s \in \{1, 2, ..., k\}$

where $j_t \in \{1, 2, ..., l\}$

Let $d^{'} = p_{i_1} ... p_{i_m}$

$d^{''} = q_{j_1} ... p_{j_n}$

$\Rightarrow d = d^{'} d^{''}$

and $d^{'}|\bar{a}, d^{''}|\bar{b}$

$(\Leftarrow) d = d^{'} d^{''}$ where $d^{'}|\bar{a}, d^{''}|\bar{b}$

$\Rightarrow \bar{a} = d^{'} k_1, \quad \bar{b} = d^{''} k_2 \quad$ some $k_1, k_2 \in \mathbb{N}$

$\Rightarrow \bar{a}\bar{b} = d^{'} d^{''} (k_1 k_2) \Rightarrow d = d^{'} d^{''} | \bar{a}\bar{b} \quad \checkmark$

If $ab = 1$, define $\bar{a}\bar{b} = 1$

Now, $h(ab) = \sum_{d|P(ab)} f(d)g(\frac{ab}{d}) = \sum_{d|\bar{a}\bar{b}} f(d)g(\frac{ab}{d})$

$= \sum_{d^{'} d^{''}|\bar{a}\bar{b}} f(d^{'} d^{''}) g(\frac{a}{d^{'}} \frac{b}{d^{''}})$

$= \sum_{d^{'}|\bar{a}} \sum_{d^{''}|\bar{b}} f(d^{'}) f(d^{''}) g(\frac{a}{d^{'}}) g(\frac{b}{d^{''}})$ $\qquad$ (since $(\bar{a}, \bar{b}) = 1 \Rightarrow (d^{'}, d^{''}) = 1$ and since $(a, b) = 1 \Rightarrow (\frac{a}{d^{'}}, \frac{b}{d^{''}}) = 1$)

$= \sum_{d^{'}|\bar{a}} f(d^{'}) g(\frac{a}{d^{'}}) \sum_{d^{''}|\bar{b}} f(d^{''}) g(\frac{b}{d^{''}})$

$= h(a)h(b) \quad \therefore \quad h$ is multiplicative

$\therefore F_h(t) = \sum_{n=1}^{\infty} h(n) t^n = (\sum_{k=0}^{\infty} h(p_1^k) t_1^k)(\sum_{k=0}^{\infty} h(p_2^k) t_2^k)$ ...

$$h(p_i^k) = \begin{cases} \sum\limits_{d|1} f(d)g(\frac{1}{d}) = 1, & \text{when k = 0.} \\ \sum\limits_{d|p_i} f(d)g(\frac{p_i^k}{d}), & \text{when k > 0} \end{cases}$$

$$= \begin{cases} 1, & \text{when k = 0.} \\ g(p_i^k) + f(p_i)g(p_i^{k-1}), & \text{when k > 0} \end{cases}$$

$$\Rightarrow \sum_{k=0}^{\infty} h(p_i^k)t_i^k = (1 + (g(p_i) + f(p_i))t_i + (g(p_i^2) + f(p_i)g(p_i))t_i^2 + ...)$$

$$= (1 + g(p_i)t_i + g(p_i^2)t_i^2 + ...) + f(p_i)t_i(1 + g(p_i)t_i + g(p_i^2)t_i^2 + ...)$$

$$= F_{i\ g}(t_i) + f(p_i)t_iF_{i\ g}(t_i)$$

$$= F_{i\ g}(t_i)(1 + f(p_i)t_i)$$

$$\Rightarrow F_h(t) = \prod_{i=1}^{\infty} F_{i\ g}(t_i)(1 + f(p_i)t_i) = F_g(t)\prod_{i=1}^{\infty}(1 + f(p_i)t_i)$$

Moreover, by noting $1 + f(p_i)t_i = 1 + f(p_i)\mu^2(p_i)t_i + f(p_i^2)\mu^2(p_i^2)t_i^2 + ...$
$= F_{i\ f\mu^2}(t_i)$, we can write $F_h(t) = F_g(t)F_{f\mu^2}(t)$
($f\mu^2$ multiplicative since it is a product of multiplicative functions)

(2)

Want $\sum\limits_{d|P(n)} f(d)\sigma(\frac{n}{d}) = 1$

Referring to ex. (1), we want $h(n) = 1$

$\therefore h = u$

Want $g = \sigma$

$\therefore$ We get, from the result of ex. (1),

$F_u(t) = F_h(t) = F_g(t)\prod\limits_{i=1}^{\infty}(1 + f(p_i)t_i) = F_\sigma(t)\prod\limits_{i=1}^{\infty}(1 + f(p_i)t_i)$

We know $F_u(t) = \frac{1}{(1-t_1)(1-t_2)\cdots}$

We know $F_\sigma(t) = \frac{1}{(1-t_1)(1-t_2)\cdots(1-p_1t_1)(1-p_2t_2)\cdots}$

$\Rightarrow f(p_i) = -p_i\ \forall i \geq 1$ (and, of course, $f(1) = 1$)

$\Rightarrow f(n) = \mu(n)I(n)$ is a possible candidate (f is multiplicative since it is the product of multiplicative functions)

Let's try it out on an example:

Let $n = p_1^2p_2$

$\Rightarrow \mu(1)(1)\sigma(p_1^2p_2) + \mu(p_1)(p_1)\sigma(p_1p_2) + \mu(p_2)(p_2)\sigma(p_1^2) + \mu(p_1p_2)(p_1p_2)\sigma(p_1) =$

$(1 + p_1 + p_2 + p_1^2 + p_1p_2 + p_1^2p_2) - p_1(1 + p_1 + p_2 + p_1p_2) - p_2(1 + p_1 + p_1^2) + p_1p_2(1 + p_1)$

$= 1$ ✓

(3)

$f : \mathbb{Z}^+ \to \mathbb{R} \ni f(n) = \chi(n\ odd)$

Let $n = ab$ where $(a, b) = 1$

Since $(a, b) = 1$, at most one of a, b is even

(case 1) a odd, b odd $\Rightarrow ab$ odd

$f(ab) = \chi(ab\ odd) = 1 = 1 \cdot 1 = \chi(a\ odd)\chi(b\ odd) = f(a)f(b)$

(case 2) a odd, b even $\Rightarrow ab$ even

$f(ab) = \chi(ab\ odd) = 0 = 1 \cdot 0 = \chi(a\ odd)\chi(b\ odd) = f(a)f(b)$

Similarly for a even, b odd

$\therefore f$ is multiplicative

First, let n odd $\Rightarrow$ d odd $\forall d|n$

$\Rightarrow f(d) = 1$

$\Rightarrow h(n) = \sum_{d|n} f(d)\phi(\frac{n}{d}) = \sum_{d|n} \phi(\frac{n}{d}) = \sum_{d|n} \phi(d) = n$ (done in class and in text) ✓

Now, let n even. Let $n = 2^l m$ where m odd and $l > 0$

We know $h(n) = (f * \phi)(n) = [t^n]F_f(t)F_\phi(t)$

$F_f(t) = (\sum_{k=0}^{\infty} f(p_1^k)t_1^k)(\sum_{k=0}^{\infty} f(p_2^k)t_2^k)\ ...$

$= (1)(1 + t_2 + t_2^2 + ...)(1 + t_3 + t_3^2 + ...)$

$= \frac{1}{(1-t_2)(1-t_3)\cdots}$

We have $F_\phi(t) = \frac{(1-t_1)(1-t_2)\cdots}{(1-p_1t_1)(1-p_2t_2)\cdots}$ (done in class)

$\therefore F_f(t)F_\phi(t) = \frac{(1-t_1)}{(1-p_1t_1)(1-p_2t_2)\cdots} = (1 - t_1)F_I(t)$

$\therefore h(p_1^{e_1}p_2^{e_2}\cdots) = [t_1^{e_1}t_2^{e_2}\cdots](1 - t_1)F_{1\ I}(t_1)F_{2\ I}(t_2)F_{3\ I}(t_3)\cdots$

$= [t_1^{e_1}t_2^{e_2}\cdots](1 - t_1)(\sum_{k=0}^{\infty} p_1^k t_1^k)F_{2\ I}(t_2)F_{3\ I}(t_3)\cdots$

$= [t_1^{e_1}t_2^{e_2}\cdots](\sum_{k=0}^{\infty}(p_1^k t_1^k - p_1^k t_1^{k+1}))F_{2\ I}(t_2)F_{3\ I}(t_3)\cdots$

Now, let $n = (2^l)(p_2^{e_2}p_3^{e_3}\cdots) = 2^l m$ with $l > 0$, m odd

What k's produce a term involving t_1^l ?

Answer: $k = l$ and $k = l - 1$ making $2^l t_1^l - 2^{l-1}t_1^l = 2^{l-1}(2 - 1)t_1^l = 2^{l-1}t_1^l$

$\therefore 2^{l-1} = [t_1^l]((1 - t_1)F_{1\ I}(t_1))$

$\therefore h(n) = [t_1^l]((1 - t_1)F_{1\ I}(t_1))[t_2^{e_2}t_3^{e_3}\cdots]F_{2\ I}(t_2)F_{3\ I}(t_3)\cdots$

$= 2^{l-1}I(m) = 2^{l-1}m = \frac{1}{2}n$ ✓