MATH 542 NUMBER THEORY
Problems to Think About #1
CH. 1, #1-3

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(1)

Same idea as in HW set, but use 3 instead of 2.

Let $\quad$ M $= 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n+1} \qquad n \geq 1$

Let $\quad$ m $= lcm(1, 3, 5, ..., 2n+1) \quad \therefore 3|m$

mM $= \frac{m}{1} + \frac{m}{3} + ... + \frac{m}{2n+1}$

contra Pf: $\quad$ Suppose M $\in \mathbb{Z} \;\Rightarrow\; 3|mM$

Let P_k be largest prime $\leq$ 2n+1

Let $s = 3^a$ be largest power of 3 in {1,3,...,2n+1}

Let t $\in$ {1,3,...,2n+1}, t $= 3^{\alpha_2}\ p_3^{\alpha_3}\ ...\ p_k^{\alpha_k}$ be UPF of t, where $p_2 = 3$

If t $<$ s, then it's clear that $\alpha_2 < a$

t $>$ s $\Rightarrow$ some $\alpha_i > 0 \quad 3 \leq i \leq k$ by maximality of 3^a

$\Rightarrow$ 2n+1 $\geq$ t $> 3^{\alpha_2}\ 3^{\alpha_i} = 3^{\alpha_2+\alpha_i} > 3^{\alpha_2}$ (Since $p_i > 3$)

$\Rightarrow \alpha_2 < a$ by maximality of 3^a

$\therefore$ If $t \neq s$, the exponent of 3 in the UPF of t is $<$ a

Now, let i $= t_i = 3^{\alpha_{i2}}\ p_3^{\alpha_{i3}}\ ...\ p_k^{\alpha_{ik}} \quad i \in$ {1,3,...,2n+1} $\quad$ be the UPF of t_i

Let $\delta_j = \max_i\{\alpha_{ij}\} \Rightarrow$ m $= 3^a\ p_3^{\delta_3}\ ...\ p_k^{\delta_k}$

$\therefore \frac{m}{s} = p_3^{\delta_3}\ ...\ p_k^{\delta_k} \Rightarrow 3 \nmid \frac{m}{s}$

For $t_i \neq s, \quad \frac{m}{t_i} = 3^{a-\alpha_{i2}}\ p_3^{\delta_3-\alpha_{i3}}\ ...\ p_k^{\delta_k-\alpha_{ik}} \quad \Rightarrow 3|\frac{m}{t_i}$ since $a > \alpha_{i2}$

$\Rightarrow 3 \nmid \frac{m}{1} + \frac{m}{3} + ... + \frac{m}{2n+1} = mM$ ($\Rightarrow\Leftarrow$)

$\therefore$ M $\notin \mathbb{Z} \quad \checkmark$

(2)

Let q_n be nth largest prime that is $\equiv 3(mod\ 4)$

Let all such primes be denoted q_i, $i \geq 1$

Let q_1,q_2,...,q_n be first n of such primes

So, $q_i = 3 + 4n_i$ some $n_i \in \mathbb{Z}$

Let X $= 1 + 2 \prod_{i=1}^{n} q_i$

$= 1 + 2 \prod_{i=1}^{n} (3 + 4n_i) = 1 + 2(3^n + 4N)$ some $N \in \mathbb{Z}$

3^n is odd $\Rightarrow 3^n = 1 + 2M$ some $M \in \mathbb{Z}$

$\Rightarrow$ X $= 1 + 2\ (1 + 2M + 4N) = 3 + 4(M + 2N) \;\Rightarrow X \equiv 3(mod\ 4)$

Clear that $2 \nmid X$ and $q_i \nmid X \quad 1 \le i \le n$

Suppose <u>all</u> prime factors of X are of the form $1 + 4\ m_i$ (which we will denote as p_i) some $m_i \in \mathbb{Z}$

$\Rightarrow$ X $= \prod p_i \equiv 1 (mod\ 4)$ ($\Rightarrow\Leftarrow$)

$\therefore$ At least one q_l with $l \ge n+1$ must divide X. $\therefore q_l \le X$

$\therefore q_{n+1} \le q_l \le X = 1 + 2 \prod_{i=1}^{n} q_i$

Claim: $q_n \le 2^{2^n}\ \forall\ n \ge 1$ (Claim determined by trial and error. i.e. so that an induction proof will work.)

Base case: n = 1 $\qquad q_1 = 3 \le 2^{2^1}$

Inductive hypothesis: Assume $q_m \le 2^{2^m}\ \ \forall m \le$ some $n \ge 1$

Consider $q_{n+1} \le 1 + 2 \prod_{i=1}^{n} q_i \le 1 + 2 \cdot 2^{2^1} \cdot 2^{2^2} \cdots 2^{2^n}$ by inductive hypothesis

$= 1 + 2^{\frac{1-2^{n+1}}{1-2}} = 1 + 2^{2^{n+1}-1} \le 2^{2^{n+1}-1} + 2^{2^{n+1}-1} = 2^{2^{n+1}}$

$\therefore$ By strong induction, $q_n \le 2^{2^n}\ \ \forall n \ge 1$ ✓

(3)

By exercise (2) $\quad q_n \le 2^{2^n}$

(case 1) $\quad x \ge 2^2 \quad$ So, there are $\ge$ n q's that are $\le 2^{2^n}$

$\Rightarrow$ there are $\ge [log_2 log_2 x]$ q's that are $\le 2^{2^{[log_2 log_2 x]}}$

$\Rightarrow$ there are $\ge log_2 log_2 x - 1$ q's that are $\le 2^{2^{log_2 log_2 x}}$

$= 2^{log_2 x} = x$

$\therefore \pi^{'}(x) \ge log_2 log_2 x - 1$ for $x \ge 2^2$

(case 2) $\quad 2 \le x < 2^2$

$q_1 = 3, q_2 = 7$

$$\therefore \pi^{'}(x) = \begin{cases} 0, & \text{if } 2 \le x < 3. \\ 1, & \text{if } 3 \le x < 2^2. \end{cases} \tag{1}$$

$0 = log_2 log_2 2^2\ - 1 > log_2 log_2 x - 1$

$\therefore \pi^{'}(x) \ge log_2 log_2 x - 1$ for $2 \le x < 2^2$

$\therefore \pi^{'}(x) \ge log_2 log_2 x - 1\ \forall x \ge 2$ ✓