Multiplicative functions and their generating functions:

A multiplicative arithmetical function is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ that satisfies $f(ab) = f(a)f(b)$ when $(a, b) = 1$, and more generally

$$f(p_1^{e_1} p_2^{e_2} \cdots) = f(p_1^{e_1}) f(p_2^{e_2}) \cdots .$$

When f is not nontrivial (not identically 0) then $f(1) = 1$.

Generating function of a non-trivial multiplicative function: Let f be a non-trivial multiplicative function and set

$$F_k(t_k) = \sum_{e=0}^{\infty} f(p_k^e) t_k^e.$$

Then

$$f(p_1^{e_1} p_2^{e_2} \cdots) = f(p_1^{e_1}) f(p_2^{e_2}) \cdots = [t_1^{e_1} t_2^{e_2} \cdots] F_1(t_1) F_2(t_2) \cdots .$$

Therefore a generating function for f is $F_f(t) = F_f(t_1, t_2, \dots) = F_1(t_1)F_2(t_2) \cdots$. Any such product with constant terms 1 in the F_k is the generating function of a multiplicative arithmetic function.

Products of generating functions:

If

$$F(t) = F(t_1, t_2, \dots) = \sum f(p_1^{e_1} p_2^{e_2} \cdots) t_1^{e_1} t_2^{e_2} \cdots = \sum_{n \geq 1} f(n) t^n$$

and

$$G(t) = G(t_1, t_2, \dots) = \sum g(p_1^{e_1} p_2^{e_2} \cdots) t_1^{e_1} t_2^{e^2} \cdots = \sum_{n \geq 1} g(n) t^n$$

then

$$F(t)G(t) = \sum f(p_1^{a_1} p_2^{a_2} \cdots) g(p_1^{b_1} p_2^{b_2} \cdots) t_1^{a_1+b_1} t_2^{a_2+b_2} \cdots =$$

$$\sum_{n \geq 1} \sum_{d|n} f(d) g(n/d) t^n.$$

This implies that if a and b are multiplicative functions with generating functions $F_a(t)$ and $F_b(t)$ then the multiplicative function c with generating function $F_a(t)F_b(t)$ is defined by

$$c(n) = \sum_{d|n} a(d) b(n/d) = \sum_{d|n} b(d) a(n/d).$$

Examples:

1. The unit function $u(n) = 1$ has generating function $F_u(t) = \frac{1}{(1-t_1)(1-t_2)\cdots}$. If $f(n)$ is multiplicative then so is

$$g(n) = \sum_{d|n} f(n/d) = \sum_{d|n} f(d)$$

and

$$F_g(t) = F_u(t)F_f(t) = \frac{F_f(t)}{(1-t_1)(1-t_2)\cdots}.$$

2. The identity function $i(n) = n$ has generating function $\frac{1}{(1-p_1t_1)(1-p_2t_2)\cdots}$. If $f(n)$ is multiplicative then so is

$$h(n) = \sum_{d|n} df(n/d) = \sum_{d|n} f(d)\frac{n}{d}$$

and

$$F_h(t) = \frac{F_f(t)}{(1-p_1t_1)(1-p_2t_2)\cdots}.$$

3. The Möbius function $\mu(n)$ defined by

$$\mu(p_1^{e_1}\cdots p_k^{e_k}) = (-1)^k\chi(e_1 = \cdots = e_k = 1)$$

has generating function

$$F_\mu(t) = (1-t_1)(1-t_2)\cdots,$$

hence is multiplicative. If f is a multiplicative function and g is defined by

$$g(n) = \sum_{d|n} f(d)$$

then we have seen by Example 1 above that

$$F_g(t) = \frac{F_f(t)}{(1-t_1)(1-t_2)\cdots} = F_u(t)F_f(t) = \frac{F_f(t)}{F_\mu(t)}.$$

This implies

$$F_f(t) = F_\mu(t)F_g(t),$$

hence

$$f(n) = \sum_{d|n} \mu(d) g(n/d) = \sum_{d|n} g(d) \mu(n/d).$$

In particular,

$$f(p^e) = g(p^e) - g(p^{e-1})$$

when p is prime and $e \geq 1$.

4. The unit characteristic function $\nu(n) = \chi(n = 1)$ has generating function $F_\nu(t) = 1$. Given that $F_\nu(t) = F_u(t) F_\mu(t)$, we have

$$\nu(n) = \sum_{d|n} \mu(d) = \sum_{d|n} \mu(n/d).$$

5. Euler's (totient) function $\phi(n)$: This is defined as the number of natural numbers $\leq n$ that are relatively prime to n. What we see in the textbook is a proof of the inclusion-exclusion formula. Working through the details, let $p_1, \ldots, p_r$ be the primes which divide n. We want to count all the numbers not divisible by any of these primes, i.e. throw away the numbers divisible at least one of these. Setting A_i equal to the numbers in $\{1, 2, \ldots, n\}$ divisible by p_i, the numbers that are divisible by some p_i are counted by $n_1 - n_2 + n_3 - \cdots$ where n_k is the sum of the sizes of the k-fold intersection intersection of sets. Since

$$|A_a \cap A_b \cap \cdots \cap A_c| = \frac{n}{p_a p_b \cdots p_z},$$

we have

$$n_k = [z^k] n (1 + \frac{z}{p_1})(1 + \frac{z}{p_2}) \cdots (1 + \frac{z}{p_r}).$$

So

$$\phi(n) = \sum_{k=0}^{r} (-1)^k [z^k] n (1 + \frac{z}{p_1})(1 + \frac{z}{p_2}) \cdots (1 + \frac{z}{p_r}) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_r}).$$

One can check that ϕ is multiplicative given this formula. Now define

$$g(n) = \sum_{d|n} \phi(d).$$

This is multiplicative. It satisfies

$$g(p^k) = \sum_{i=0}^{k} \phi(p^i) = 1 + (p-1) + (p^2 - p) + \cdots + (p^k - p^{k-1}) = p^k,$$

hence $g(n) = n$ for all n. Therefore

$$\sum_{d|n} \phi(d) = n.$$

To obtain a generating function for $\phi(n)$, note that

$$F_i(t) = F_g(t) = F_u(t) F_\phi,$$

hence

$$F_\phi(t) = F_\mu(t) F_i(t) = \frac{(1-t_1)(1-t_2)\cdots}{(1-p_1 t_1)(1-p_2 t_2)\cdots}.$$

6. Möbius Inversion: Let $f : \mathbb{R} \to \mathbb{R}$ be given and define

$$g(x) = \sum_{n \le x} f(x/n),$$

summing over positive integers. Then

$$\sum_{n \le x} \mu(n) g(x/n) = \sum_{n \le x} \mu(n) \sum_{m \le x/n} f(x/mn) = \sum_{n \le x} \mu(n) \sum_{mn \le x} f(x/mn) =$$

$$\sum_{l \le x} f(x/l) \sum_{m|l} \mu(l/m) = \sum_{l \le x} f(x/l) \nu(l) = f(x).$$

Conversely, if we define

$$f(x) = \sum_{n \le x} \mu(n) g(x/n)$$

then

$$\sum_{n \le x} f(x/n) = \sum_{n \le x} \sum_{k \le x/n} \mu(n) g(x/kn) = \sum_{n \le x} \sum_{kn \le x} \mu(n) g(x/kn) = \sum_{l \le x} g(x/l) \sum_{m|l} \mu(l/m) =$$

$$\sum_{l \le x} g(x/l) \nu(l) = g(x).$$

When a multiplicative function is used to define the other this way then the second function is also multiplicative, and we obtain

$$g(n) = \sum_{d|n} f(d)$$

if and only if

$$f(n) = \sum_{d|n} \mu(d) g(n/d).$$

We already derived this by the method of generating functions above.

7. Applying Möbius inversion to the functions

$$\tau(n) = \sum_{d|n} 1,$$

$$\sigma(n) = \sum_{d|n} d,$$

$$n = \sum_{d|n} \phi(d),$$

we obtain

$$1 = \sum_{d|n} \mu(d) \tau(\frac{n}{d}),$$

$$n = \sum_{d|n} \mu(d) \sigma(\frac{n}{d}),$$

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

The identities above also follow from $F_u = F_\tau F_\mu$, $F_i = F_\sigma F_\mu$, $F_\phi = F_i F_\mu$.

8. Summary of generating functions:

$\mu(n)$: $F_\mu = (1 - t_1)(1 - t_2) \cdots$

$\nu(n) = \chi(n = 1) = \sum_{d|n} \mu(n)$: $F_\nu = 1$

$u(n) = 1$: $F_u = \frac{1}{(1-t_1)(1-t_2)\cdots}$

$i(n) = n$: $F_i = \frac{1}{(1-p_1 t_1)(1-p_2 t_2)\cdots}$

$\tau(n) = \sum_{d|n} 1 = \sum_{d|n} u(d)$: $F_\tau = F_u^2 = \frac{1}{(1-t_1)^2(1-t_2)^2\cdots}$

$\sigma(n) = \sum_{d|n} d = \sum_{d|n} i(d)$: $F_\sigma = F_u F_i = \frac{1}{(1-t_1)(1-t_2)\cdots(1-p_1 t_1)(1-p_2 t_2)\cdots}$

$\phi(n)$: $F_\phi = \frac{(1-t_1)(1-t_2)\cdots}{(1-p_1 t_1)(1-p_2 t_2)\cdots} = F_\mu F_i$.

9. The Riemann zeta-function. Take any generating function $F(t) = F(t_1, t_2, \dots) = F_1(t_1)F_2(t_2)\cdots$ for a multiplicative function f. Making the substitution $t_i \mapsto \frac{1}{p_i^s}$ where s is a complex number yields an infinite product. For example, recall that we have

$$F_u(t) = \frac{1}{(1-t_1)(1-t_2)\cdots} = \sum_{e_1,e_2,e_3,\dots \geq 0} t_1^{e_1} t_2^{e_2} t_3^{e_3} \cdots.$$

Hence

$$F_u(s) = F_i(1/p_1^s, 1/p_2^s, \dots) = \sum_{e_1,e_2,e_3,\dots \geq 0} \frac{1}{p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This is called the Riemann zeta-function $\zeta(s)$. In particular,

$$\zeta(2) = \prod_p \frac{1}{1-(1/p^2)} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We will derive this evaluation this shortly.

More generally, if $F_f(t) = \sum_{n=1}^{\infty} f(n) t^n$ then

$$F_f(s) = F_f(1/p_1^s, 1/p_2^s, \dots) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Examples:

1. $F_\mu(s) = \frac{1}{F_u(s)}$. This implies

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

2. $F_\tau(s) = F_u(s)^2$. This implies

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \zeta(s)^2.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{\pi^4}{36}.$$

3. $F_i(s) = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1)$.
4. $F_\sigma(s) = F_i(s)F_u(s)$. This implies

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s-1)\zeta(s).$$

5. $F_\phi(s) = F_\mu(s)F_i(s)$. This implies

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

6. For arbitrary functions $f : \mathbb{Z}^+ \to \mathbb{R}$ and $g : \mathbb{Z}^+ \to \mathbb{R}$ we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{n=1}^{\infty} \frac{g(n)}{n_s} = \sum_{a,b \geq 1} \frac{f(a)g(b)}{(ab)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} f(d)g(n/d)$$

assuming the expressions converge.