Problems to think about (Chapter 6)

1. Let a and b be positive integers such that a is not divisible by b. Write $\frac{a}{b} = [a_0, a_1, \dots, a_n]$. Prove that $|aq_{n-1} - bp_{n-1}| = (a, b)$.

2. Let θ be an irrational number with partial quotients a_n and convergents $\frac{p_n}{q_n}$. Prove that

$$\frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}},$$

hence

$$\theta = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k q_{k-1}}.$$

3. Given two sequences of integers $a_0, a_1, a_2, \dots$ and $b_1, b_2, b_3, \dots$, define the two sequences $p_0, p_1, p_2, \cdots$ and $q_0, q_1, q_2 \cdots$ via $p_0 = a_0$, $p_1 = a_0 a_1 + b_1$, $q_0 = 1$, $q_1 = a_1$, and for $n \geq 2$,

$$p_n = a_n p_{n-1} + b_n p_{n-2}$$

and

$$q_n = a_n q_{n-1} + b_n q_{n-2}.$$

a. Prove that for $n \geq 1$,

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & b_1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & b_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}.$$

b. Assume $a_i, b_i > 0$ for $i \geq 1$. Prove that if we define the symbol $[a_0, \dots, a_n; b_1, \dots, b_n]$ recursively by $[a_0; -] = a_0$ and

$$[a_0, a_1, \dots, a_n; b_1, b_2, \dots, b_n] = a_0 + b_1/[a_1, a_2, \dots, a_n; b_2, \dots, b_n]$$

for $n \geq 1$ then for all n we have

$$[a_0, \dots, a_n; b_1, \dots, b_n] = \frac{p_n}{q_n}.$$

c. Prove that $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1}\frac{\prod_{i=1}^n b_i}{q_n q_{n-1}}$ for all $n \geq 1$, hence

$$\frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} (-1)^{k-1}\frac{\prod_{i=1}^k b_i}{q_k q_{k-1}}.$$

d. Given $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, write $c_k = \frac{q_k q_{k-1}}{\prod_{i=1}^k b_k}$ for $k \geq 1$. Prove the following identities:

(1) For $k \geq 1$,

$$\frac{c_{k+1} - c_k}{c_k} = \frac{a_{k+1}}{b_{k+1}}\frac{q_k}{q_{k-1}}.$$

(2) Substituting (1) twice into

$$\frac{q_k}{q_{k-1}} = a_k + b_k\frac{q_{k-2}}{q_{k-1}},$$

prove

$$\frac{a_{k+1}a_k}{b_{k+1}} = \frac{(c_{k+1} - c_k)(c_k - c_{k-1})}{c_k^2}$$

for $k \geq 2$.

e. Let $x_1 < x_2 < x_3 < \cdots$ be an arbitrary sequence of positive integers that satisfies $x_k \to \infty$. Show that setting $a_0 = 0$, $a_1 = x_1$, $b_1 = 1$, and for $k \geq 2$, $a_k = x_k - x_{k-1}$ and $b_k = x_{k-1}^2$, we obtain

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}} = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n; b_1, \ldots, b_n] = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{x_k}.$$

f. Using (e), prove that

$$\ln(2) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cdots}}}}}$$