Problems to think about (Chapter 5)

1. Let $f = ax^2 + bxy + cy^2$ be a non-reduced binary quadratic form with $a > 0$ and $d = b^2 - 4ac < 0$. Then f falls into exactly one of the following categories: (1) $a > c$. (2) $a = c$ and $b < -a$. (3) $a = c$ and $-a \leq b < 0$. (4) $a = c$ and $b > a$. (5) $a < c$ and $b < -a$. (6) $a < c$ and $b = -a$. (7) $a < c$ and $b > a$. Prove that the following actions, applied a finite number of times, yield an equivalent reduced form. In the statements below, $[[x]]$ represents the largest integer strictly less than x.

If f is in category (1) or (3), replace by $f = cx^2 - bxy + ax^2$.

If f is in categories (2) or (5), replace by $f = ax^2+(b+2ka)xy+(ak^2+bk+c)y^2$ where $k = [[-b/a]]$.

If f is in category (4) or (7), replace by $f = ax^2+(b-2ka)xy+(ak^2-bk+c)y^2$ where $k = [[b/a]]$.

If f is in category (6), replace by $f = ax^2 + (b + 2a)xy + (a + b + c)y^2 = ax^2 + axy + cy^2$.

2. Assuming you have successfully completed Problem 1, use the information it contains to write a Mathematica program that inputs $\{a, b, c\}$ where $a > 0$ and $b^2-4ac < 0$ and outputs $\{A, B, C\}$ where $Ax^2+Bxy+Cy^2$ is reduced and equivalent to $ax^2 + bxy + cy^2$. The program should output all intermediate forms between $ax^2 + bxy + cy^2$ and $Ax^2 + Bxy + Cy^2$.

3. $f(x, y) = 12345x^2 + 18309xy + 6789y^2$. Find a unimodular matrix U such that $f \circ U$ is reduced. Compute the minimum positive value of $f(x, y)$ given $x, y \in \mathbb{Z}$, and identify the corresponding values of x and y.

4. Write a Mathematica program that inputs an odd prime p and outputs $\{x, y, z, w\}$ satisfying $p = x^2+y^2+z^2+w^2$ using the infinite descent procedure outlined in class.

5. Write a Mathematica program that inputs $\{\{p_1, e_1\}, \{p_2, e_2\}, \ldots, \{p_r, e_r\}\}$ and outputs $\{x, y, z, w\}$ satisfying $p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} = x^2 + y^2 + z^2 + w^2$ where the p_i are prime numbers and the e_i are positive integers. Use the program in Problem 4 as a subroutine.

6. The Mathematica expression `FactorInteger[n]` outputs the prime factorization of n in the form $\{\{p_1, e_1\}, \{p_2, e_2\}, \ldots, \{p_r, e_r\}\}$. Using this, express the number 123456789 as a sum of four squares.

7. Devise an infinite descent algorithm for expressing a prime p of the form $4k + 1$ as a sum of two squares and implement it in Mathematica.