Inclusion-Exclusion Sum and Product

Let f be a function. We wish to evaluate

$$\sum_{\substack{1\le a\le n\\(a,n)=1}} f(a).$$

Let the prime factorization of n be $n = p_1^{e_1}\cdots p_r^{e_r}$. For each $d \le n$ let $A_d = \{a \le n : d|a\}$. By the inclusion-exclusion formula,

$$\sum_{\substack{1\le a\le n\\(a,n)>1}} f(a) = \sum_{k=1}^{r} (-1)^k n_k$$

where

$$n_k = \sum_{1\le i_1<i_2<\cdots<a_k\le r}\ \sum_{a\in A_{p_{i_1}}\cap A_{p_{i_2}}\cdots\cap A_{p_{i_k}}} f(a) =$$

$$\sum_{1\le i_1<i_2<\cdots<a_k}\ \sum_{a\in A_{p_{i_1}p_{i_2}\cdots p_{i_k}}} f(a).$$

This implies

$$\sum_{\substack{1\le a\le n\\(a,n)>1}} f(a) = -\sum_{d|P}\mu(d)\sum_{a\in A_d} f(a) + \sum_{a=1}^{n} f(a).$$

Hence

$$\sum_{\substack{1\le a\le n\\(a,n)=1}} f(a) = \sum_{d|P}\mu(d)\sum_{a\in A_d} f(a).$$

where

$$P = p_1p_2\cdots p_r.$$

Corollary:

$$\prod_{\substack{1\le a\le n\\(a,n)=1}} f(a) = \prod_{d|P}\left(\prod_{a\in A_d} f(a)\right)^{\mu(d)}.$$

Lemma:
$$\sum_{d|P} \mu(d) d^k = (1-p_1^k)(1-p_2^k)\cdots(1-p_r^k).$$

Examples:

1. $f(a) = a$. Then
$$\sum_{a \in A_d} f(a) = n^2/2d + n/2,$$
$$\sum_{\substack{1 \le a \le n \\ (a,n)=1}} f(a) = \frac{n^2}{2}(1-1/p_1)\cdots(1-1/p_r) = \frac{n\phi(n)}{2}.$$

2. $f(a) = a^3$. Then $\sum_{a \in A_d} f(a) = (dn^2)/4 + n^3/2 + n^4/(4d)$,
$$\sum_{\substack{1 \le a \le n \\ (a,n)=1}} f(a) = \frac{n^2}{4}(1-p_1)\cdots(1-p_r) + \frac{n^4}{4}(1-1/p_1)\cdots(1-1/p_r) =$$
$$\frac{\phi(n)}{4}((-1)^r p_1 \cdots p_r n + n^3).$$

3. $f(a) = a$. Then $\prod_{a \in A_d} f(a) = d^{n/d}(n/d)!$, therefore
$$\prod_{\substack{1 \le a \le n \\ (a,n)=1}} a = \prod_{d|P} \left(d^{n/d}(n/d)!\right)^{\mu(d)} = \prod_{d|P} \left((n/d)^d d!\right)^{\mu(n/d)} =$$
$$n^{\sum_{d|n} d\mu(n/d)} \prod_{d|n} (d!/d^d)^{\mu(n/d)} = n^{\phi(n)} \prod_{d|n} (d!/d^d)^{\mu(n/d)}.$$