An Infinite Descent Problem

Let p be a prime number. Then $a^3 + pb^3 + p^2c^3 = pabc$ has no non-trivial integer solutions.

Proof: Suppose there is a solution with $a \neq 0$. We have $p|a$. Substituting $a = p\alpha$ we obtain

$$p^3\alpha^3 + pb^3 + p^2c^3 = p^2\alpha bc.$$

Dividing by p we obtain

$$p^2\alpha^3 + b^3 + pc^3 = p\alpha bc.$$

Substituting $b = p\beta$ we obtain

$$p^2\alpha^3 + p^3\beta^3 + pc^3 = p^2\alpha\beta c.$$

Dividing by p we obtain

$$p\alpha^3 + p^2\beta^3 + c^3 = p\alpha\beta c.$$

Substituting $c = p\gamma$ we obtain

$$p\alpha^3 + p^2\beta^3 + p^3\gamma^3 = p^2\alpha\beta\gamma.$$

Dividing by p we obtain

$$\alpha^3 + p\beta^3 + p^2\gamma^3 = p\alpha\beta\gamma.$$

So if there is a solution with a non-zero value of a then there is another solution with a replaced by a/p. This yields solutions with a non-zero integer a/p^k for all $k \geq 1$, which is not possible. Hence any solution must satisfy $a = 0$. If there is a solution with $b \neq 0$, then

$$pb^3 + p^2c^3 = 0,$$

$$b^3 + pc^3 = 0.$$

Substituting $b = p\beta$ we obtain

$$p^3\beta^3 + pc^3 = 0,$$

$$p^2\beta^3 + c^3 = 0.$$

Substituting $c = p\gamma$ we obtain

$$p^2\beta^3 + p^3\gamma^3 = 0,$$

$$\beta^3 + p\gamma^3 = 0.$$

This yields another infinite descent, so $b = 0$. Hence $a = b = c = 0$ is the only solution.

More generally, let p be a prime number. Then $a_1^n + pa_2^n + \cdots + p^{n-1}a_n^n = pa_1a_2\cdots a_n$ has no non-trivial integer solutions.