Theorem: *Let M be a non-negative square matrix pattern with positive diagonal. If the associated diagraph D has an even cycle then there is a singular matrix A with sign-pattern M.*

Proof: Choose a cycle of length $2k$ in D and assign $a > 0$ to each of the corresponding entries of A. Assign 1 to the diagonal entries. Assign $b > 0$ to all the other positive entries. The determinant of A is

$$1 - a^{2k}(1 + f(b)) + g_b(a) + h(b),$$

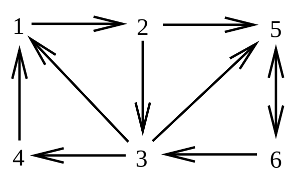
where $f(x)$, $g_b(x)$, and $h(x)$ are polynomials with no constant term and $\deg g_b(x) < 2k$. Choose $b > 0$ sufficiently small so that $1 + f(b) > 0$ and $1 + h(b) > 0$. Having chosen b, set $k_b(x) = -x^{2k}(1 + f(b)) + g_b(x) + 1 + h(b)$. The determinant of A is $k_b(a)$. Since $k_b(x)$ is a non-constant polynomial with negative leading coefficient and positive constant term, a sufficiently large $a > 0$ forces $k_b(a) < 0$, and a sufficiently small $a > 0$ forces $k_b(a) > 0$. Hence by the Intermediate Value Theorem there is a positive value of a such that $k_b(a) = 0$.

Example: Let

$$M = \begin{pmatrix} + & + & 0 & 0 & 0 & 0 \\ 0 & + & + & 0 & + & 0 \\ + & 0 & + & + & + & 0 \\ + & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & + & 0 & + & + \end{pmatrix}.$$

The associated digraph is

$D =$



.

Choosing the 4-cycle $1 \to 2 \to 3 \to 4 \to 1$, we create the matrix

$$A = \begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & b & 0 \\ b & 0 & 1 & a & b & 0 \\ a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & b & 0 & b & 1 \end{pmatrix}.$$

We have

$$\det\, A = 1 - a^4 + a^2 b - b^2 + a^4 b^2 + b^3 - a^2 b^3 - a^3 b^3 + a b^4 =$$

$$1 - a^4(1 - b^2) + (-b^3 a^3 + (-b^3 + b)a^2 + b^4 a) + b^3 - b^2.$$

Hence we can identify

$$f(x) = -x^2,$$
$$g_b(x) = -b^3 x^3 + (-b^3 + b)x^2 + b^4 x,$$
$$h(x) = x^3 - x^2.$$

We have $1 + f(0.5) = 0.75$ and $1 + h(0.5) = 0.875$, so we set $b = 0.5$. We now have

$$k_{0.5}(x) = 0.875 + 0.0625x + 0.375x^2 - 0.125x^3 - 0.75x^4.$$

We have $k_{0.5}(1) = 0.4375$ and $k_{0.5}(2) = -10.5$, so for some value of $a \in (1, 2)$ we have $k_{0.5}(a) = 0$. Using this value of a, the matrix

$$A = \begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0.5 & 0 \\ 0.5 & 0 & 1 & a & 0.5 & 0 \\ a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 & 1 \end{pmatrix}$$

has determinant zero. (The approximate value of a is 1.1354073181108784. Using this, the determinant is -1.47773×10^{-16}.)