Vector Spaces Lecture

Abelian group: a set G and an addition operator on the set. Addition is associative and commutative. There must be an identity element and each element must have an additive inverse.

Examples: Integers, rationals, reals, complexes, modulus groups.

Field: an abelian group with a multiplication operator on the set. Multiplication is associative and commutative. There must be an identity element and each non-zero element must have an additive inverse.

Examples: rationals, reals, complexes, prime-modulus groups.

Vector Space over a field: an abelian group with a scalar multiplication defined. the scalar multiplication must satisfy $(rs) \cdot v = r \cdot (s \cdot v)$, $1 \cdot v = v$, $r \cdot (v + w) = (r \cdot v) + (s \cdot v)$.

Examples: F^n where F is a field, solutions in F^n to a system of homogeneous equations, real-valued functions defined on a interval, continuous real-valued functions defined on an interval, polynomial functions, polynomial functions that send a subset of values to zero.

Subspaces. Examples: see preceeding paragraph.

Sums of subspaces.

Direct sums of subspaces: each element is a unique sum of elements in the summand subspaces.

Properties of polynomial functions of a field: Degree of a polynomial. Division algorithm. If a is a root of $p(x)$ then $p(x) = (x - a)q(x)$. Therefore a polynomial can only have a finite number of roots. Therefore the coefficients of a polynomial function of an infinite field are unique. Not true in the case of a prime-modulus field since $x^p = x$ where p is the modulus of the field. Every complex polynomial function has a root (Math 641), hence every complex polynomial function can be factored into linear factors. If z is a complex root of $p(x) \in \mathbb{R}[x]$ then so is $\overline{z}$. Hence the complex roots appear in complex conjugate pairs. Hence a real polynomial can be factored into linear and quadratic factors.

A direct sum example: real polynomials can be decomposed into U_e, and U_o where U_e is the span of $\{1, x^2, x^4, \ldots\}$ and U_o is the span of $\{x, x^3, x^5, \ldots\}$. This is not true when the field is $\mathbb{Z}_2$ since $x = x^2$ is a non-zero vector in $U_o \cap U_e$.

Abstract properties of vector spaces: see the propositions and the exercises.