Linear Maps

Definition

Examples: Identity, differentiation, integration, multiplication by x^2, backward shift

Algebra: Vector space with an associative product of vectors. Moreover $X(aY) = a(XY)$.

The algebra of linear operators from V to V: multiplication given by function composition. Notation: $\mathcal{L}(V)$.

Nullspace and range of a linear operator. These are subspaces of domain and codomain. Dimensions are called rank and nullity.

Theorem: A linear operator is injective iff its nullspace is trivial.

Rank-Nullity Theorem: Given a linear operator $T : V \to W$,

$$\dim V = \text{rank} + \text{nullity}.$$

Proof: Find a basis $\{n_1, \ldots, n_a\}$ for the nullspace and a basis $\{y_1, \ldots, y_b\}$ for the range. Pull the range basis back to the codomain, yielding $\{x_1, \ldots, x_b\}$ where $Tx_i = y_i$. Verify that a basis for V is $\{n_1, \ldots, n_a, x_1, \ldots, x_b\}$: Let $v \in V$. Then $Tv = \sum_i a_i y_i$, therefore $T \sum_i a_i x_i = Tv$, therefore $\sum_i a_i x_i - v \in$ nullspace, therefore $\sum_i a_i x_i - v = \sum_i b_i n_i$, hence v is spanned by these vectors. Moreover $\sum_i a_i n_i + \sum_i b_i x_i = 0$ implies $\sum_i b_i y_i = 0$, hence each $b_i = 0$, hence each $a_i = 0$.

Corollary: A linear map is injective if and only if the dimension of its domain is equal to the dimension of its range. A linear map is surjective if and only if the dimension of its range is equal to the dimension of its codomain. A linear map is bijective if and only if domain, range, and codomain have the same dimension. For a linear map $T : V \to V$, if T is injective then $\dim V = \text{rank}$, hence domain, range, and codomain have the same dimension, hence T is bijective. The same is true of a surjective map $T : V \to V$. Hence injective, surjective, and bijective are all equivalent for $T \in \mathcal{L}(V, V)$. Note that the inverse of a bijective linear map is also linear.

Isomorphic Vector Spaces: A pair of vector spaces with a bijective linear map from one to the next. Two vector spaces are isomorphic if and only if they have the same dimension. To construct an isomorphism, map a basis to another basis.

Counting Solutions to a linear system of equations: If A is an $m \times n$ matrix and x is an $n \times 1$ column vector, then Ax is the $m \times 1$ column vector consisting of the linear combination of the columns of A using the entries of x. Hence any linear system of equations can be expressed in the form $Ax = b$. This gives rise to a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ via $Tx = Ax$. The range of T is the span of the columns of A, hence rank $\leq n$. So if $m > n$ the map is not surjective. In other words, too many equations implies some solutions to $Ax = b$ don't exist. Also, if $m < n$ then the map can't be injective, so there must be some non-trivial solution to $Ax = 0$. (Or: there are more column vectors than the dimension of the space they live in, so they are linearly dependent.) Solutions to $Ax = b$ always exist and are unique (both) if and only of $m = n$.

The matrix of a linear map: Let $T : V \to W$ be linear with V-basis $\{v_1, \ldots, v_m\}$ and W-basis $\{w_1, \ldots, w_n\}$. Suppose that for each v_j,

$$Tv_j = \sum_i a_{ij} w_i.$$

We say that $m(T) = (a_{ij})$. For a vector $v = \sum_i x_i v_i \in V$ we write

$$m(v) = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

For a vector $w = \sum_i y_i w_i \in W$ we write

$$m(w) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

With these conventions we have

$$m(Tv) = m(T)m(v).$$

Proof: Write $m(v) = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ and $m(Tv) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Then

$$Tv = T(\sum_j x_j v_j) = \sum_j x_j T v_j = \sum_j x_j (\sum_i a_{ij} w_i) = \sum_i (\sum_j a_{ij} x_j) w_i,$$

which implies

$$y_i = \sum_j a_{ij} x_j$$

for each i. In other words,

$$m(Tv) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = (a_{ij}) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = m(T)m(v).$$

The matrix of a composition of linear maps: Let $T : U \to V$ and $S : V \to W$ be linear maps with U-basis $\{u_i\}$, V-basis $\{v_i\}$, and W-basis $\{w_i\}$. Then $ST : U \to W$ satisfies $m(ST) = m(S)m(T)$.

Proof: $m(STu) = m(S)m(Tu) = m(S)m(T)m(u)$ for all u. In particular, $m(STu_i) = m(S)m(T)m(u_i) = m(S)m(T)e_i =$ column i of $m(S)m(T)$. This implies that $m(ST) = m(S)m(T)$.

The dimension of $\mathcal{L}(V, W)$: There is an isomorphism between $\mathcal{L}(V, W)$ and the appropriate size matrices which represent these linear maps. This yields

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W.$$