Finite-Dimensional Vector Spaces Lecture

Terminology: Linear combination, span, finite-dimensional, infinite-dimensional, linearly independent, linearly dependent, basis.

Lemma: Every finite-dimensional vector space has a basis.

Proof: Keep discarding dependent vectors without changing the span until no longer possible

Lemma: Let V be spanned by n vectors over F. Then any $m > n$ vectors in V are linearly dependent.

Proof: By induction on n. For $n = 1$, if V is spanned by $\{v_1\}$ then any av_1 and bv_1 are linearly dependent using the coefficients $b, -a$ or $1, 1$. Assume whenever V is spanned by n vectors then any $m > n$ are linearly dependent. Now suppose V is spanned by $\{v_1, \ldots, v_{n+1}\}$. Let $u_1, \ldots, u_m$ be arbitrary where $m > n + 1$. Let W be the span of $\{v_1, \ldots, v_n\}$. If each $u_i \in W$ then $u_1, \ldots, u_m$ are linearly dependent by the induction hypothesis. Otherwise, there is one of them, wlog u_m, that has a non-zero coefficient of v_{n+1}. We can find coefficients $a_1, \ldots, a_{m-1}$ such that for each $i < m$, $u_i - a_i u_m$ lies in W, and hence $u_1 - a_1 u_m, \ldots, u_{m-1} - a_{m-1} u_m$ are linearly dependent by the induction hypothesis since $m - 1 > n$. Examining the dependency relationship we see that u_1 through u_m are dependent.

Corollary: Every two bases for a finite-dimensional vector space have the same size.

Proof: If there were two different size bases, the smaller one would make the larger one dependent.

Definition: The dimension of a finite-dimensional vector space is the size of any basis.

Theorem: Let V be a finitely generated vector space over a field $\mathbb{F}$ and let X be an arbitrary set of linearly independent vectors in V. Then there is a basis for V which contains X as a subset.

Proof: If there is a vector outside the span of X, add it as an additional linearly independent vector. Keep on going. Process must halt at a basis since there are at most n linearly independent vectors.

Theorem: Let V be an n-dimensional vector space. Then any n linearly independent vectors form a basis.

Proof: There is a basis of size n that contains this set, so this set is already a basis.

Theorem: Let V be an n-dimensional vector space. Then any spanning set of size n must be a basis.

Proof: Any spanning set of size n must contain a basis of n vectors, so the spanning set is already a basis.

Dimension of $W < V$: Expand basis of W to basis of V.

Theorem: If V is finite-dimensional and $V = U_1 \bigoplus \cdots \bigoplus U_n$ then $\dim V = \dim U_1 + \cdots + \dim U_n$.

Proof: A basis for V is the union of bases for each U_i.

Theorem: If V is finite-dimensional, $V = U_1 + \cdots + U_n$, and $\sum_i \dim U_i = \dim V$ then $V = \sum_i \bigoplus U_i$.

Proof: The union of the U_i bases spans V and has size less than or equal to the dimension of V, hence contains a basis for V which must be the entire union. This implies that the union is linearly independent and that sums are unique.

Dimension of $U_1 + U_2$: First find basis for $U_1 \cap U_2$: $\{x_i\}$. Let basis for U_1 be $\{x_i, y_j\}$ and let basis for U_2 be $\{x_i, z_k\}$. Then $\{x_i, y_j, z_k\}$ is basis for $U_1 + U_2$: must show span and linear independence.

Notation: The typical vector in the span of $\{v_1, \ldots, v_n\}$ is V. Now we can say that $X + Y = 0$ implies $X = Y = 0$ and $X + Z = 0$ implies $X = Z = 0$.

Span: The typical vector is $u_1 + u_2 = (X_1 + Y) + (X_2 + Z) = (X_1 + X_2) + Y + Z$.

Independence: Suppose $X + Y + Z = 0$. Then $Z = -X - Y \in U_1 \cap U_2$. This implies $Z = X'$. Therefore $X + X' + Y = 0$, which forces $Y = 0$. So now we have $X + Z = 0$, therefore $X = Z = 0$. Hence $U_1 + U_2 = \text{span}(x_i) \bigoplus \text{span}(y_j) \bigoplus \text{span}(z_k)$, which implies the dimension result.