Exam 4 Solutions Math 447/547 Spring 2013

Due Thursday, May 9, noon

Be thorough!

Review of trace: Let V be an n-dimensional vector space, let $T \in \mathcal{L}(V)$, and let $A = (a_{ij})$ be any matrix representation of T. Then

$$\text{trace}(T) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Review of determinant: Let V be an n-dimensional vector space, let $T \in \mathcal{L}(V)$, and let $A = (a_{ij})$ be any matrix representation of T. Then

$$\det(T) = \sum_{n \in \mathcal{S}_n} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where for each permutation $\sigma \in \mathcal{S}_n$,

$$\text{sgn}(\sigma) = (-1)^{n-c(\sigma)}$$

and $c(\sigma)$ is the number of disjoint cycles in the cycle-decomposition of σ.

Notation: Let F be a field. The standard basis for $M_{n,n}(F)$ is

$$\{E_{ij} : 1 \leq i, j \leq n\},$$

where E_{ij} is the matrix with 1 in row i, column j and zeros elsewhere.

Exam Problems:

1. Let $S, T \in \mathcal{L}(V)$ be linear maps on an n-dimensional space V. In class we proved that $\text{trace}(ST) = \text{trace}(TS)$. Now consider $T_1, T_2, \ldots, T_k \in \mathcal{L}(V)$.

(a) (16 points) Prove

$$\text{trace}(T_1 T_2 \cdots T_k) = \text{trace}(T_2 T_3 \cdots T_k T_1) = \text{trace}(T_3 T_4 \cdots T_k T_1 T_2) = \cdots.$$

(b) (16 points) Show that $\text{trace}(T_1 T_2 T_3) \neq \text{trace}(T_2 T_1 T_3)$ using an appropriate example.

Solutions:

(a) Write $X = T_1$ and $Y = T_2 \cdots T_k$. Then $\text{trace}(XY) = \text{trace}(YX)$, which implies the first equality. The remaining equalities are all implied by the first one.

(b) Fix an arbitrary basis B for V and let $T_1 = X_{12}$, $T_2 = X_{23}$, $T_3 = X_{31}$. Then the matrix representation for $T_1T_2T_3$ is $E_{12}E_{23}E_{33} = E_{11}$ and the matrix representation for $T_2T_1T_3$ is $E_{23}E_{12}E_{11} = 0$. Hence $\text{trace}(T_1T_2T_3) = 1$ and $\text{trace}(T_2T_1T_3) = 0$.

2. Let V be an n-dimensional inner-product space with inner-product $\langle \bullet, \bullet \rangle$. An inner-product on $\mathcal{L}(V)$ itself is given by

$$\langle R, S \rangle' = \text{trace}(RS^*)$$

for all $R, S \in \mathcal{L}(V)$ (see Problem 18, Chapter 10). Now let T be one particular linear map in $\mathcal{L}(V)$. We will define the linear map $f_T : \mathcal{L}(V) \to \mathcal{L}(V)$ by $f_T(X) = TXT^*$ for all $X \in \mathcal{L}(V)$. Follow the steps below to express the singular values of f_T in terms of the singular values of T.

(a) (16 points) Let $F_T : \mathcal{L}(V) \to \mathcal{L}(V)$ be the linear map defined by $F_T(X) = T^*XT$ for all $X \in \mathcal{L}(V)$. Prove that $(f_T)^* = F_T$ with respect to the inner-product $\langle \bullet, \bullet \rangle'$.

(b) (16 points) Let B be an arbitrary orthonormal basis for V and let $X_{ij} \in \mathcal{L}(V)$ have matrix representation E_{ij} for each $1 \le i, j \le n$ with respect to B. Prove that the set

$$\mathcal{X}_B = \{X_{ij} : 1 \le i, j \le n\}$$

forms an orthonormal basis for $\mathcal{L}(V)$ with respect to $\langle \bullet, \bullet \rangle'$.

(c) (16 points) Assuming that the n singular values of T are $s_1, \ldots, s_n$, find the n^2 singular values of f_T. Hint: You will find the Spectral Theorem useful for finding an orthonormal basis of eigenvectors for V with respect to T^*T and $\langle \bullet, \bullet \rangle$, and you will find the set $\mathcal{X}_B$ useful for finding an orthonormal basis of $\mathcal{L}(V)$ with respect to $(f_T)^* f_T$ and $\langle \bullet, \bullet \rangle'$ and a suitable choice of B.

Solutions:

(a) $\langle f_T(X), Y \rangle' = \langle TXT^*, Y \rangle' = \text{trace}(TXT^*Y^*) = \text{trace}(XT^*Y^*T) = \text{trace}(X(T^*YT)^*) = \langle X, T^*YT \rangle' = \langle X, F_T(Y) \rangle'$ for all $X, Y \in \mathcal{L}(V)$. The unique linear operator F_T with this property is $(f_T)^*$.

(b) It is easy to verify that $E_{ij}E_{ab} = \delta_{ja}E_{ib}$ and $\text{trace}(E_{ij}) = \delta_{ij}$ and $E_{ij}^* = E_{ji}$, therefore

$$\langle X_{ij}, X_{ab} \rangle' = \text{trace}(X_{ij}X_{ab}^*) = \text{trace}(E_{ij}E_{ba}) = \text{trace}(\delta_{jb}E_{ia}) = \delta_{jb}\delta_{ia}.$$

Hence the inner product of $X_{ij}X_{ab}$ is equal to 1 if $X_{ij} = X_{ab}$ and is equal to 0 if $X_{ij} \neq X_{ab}$. This implies that the collection of X_{ij} is orthonormal in $\mathcal{L}(V)$.

(c) We need only find the eigenvalues of the associated positive linear operator, then produce the positive square root of each. We have

$$(f_T)^* f_T X = (f_T)^*(TXT^*) = (T^*T)X(T^*T).$$

Since T^*T is a positive linear operator, V has an orthonormal basis of eigenvectors B with respect to which T^*T has matrix representation $\text{diag}(s_1^2, \dots, s_n^2)$. Denoting by A the matrix representation of X with respect to the same basis, an eigenvector of $f_T^* f_T$ will have matrix representation A such that

$$\text{diag}(s_1^2, s_2^2, \dots, s_n^2) \cdot A \cdot \text{diag}(s_1^2, s_2^2, \dots, s_n^2)A = \lambda A$$

for some $\lambda \in F$. Note that

$$\text{diag}(s_1^2, s_2^2, \dots, s_n^2) \cdot E_{ij} \cdot \text{diag}(s_1^2, s_2^2, \dots, s_n^2) = s_i^2 s_j^2 E_{ij}$$

for each i and j. This implies

$$(T^*T)X_{ij}(T^*T) = s_i^2 s_j^2 X_{ij}$$

for each i and j, i.e.

$$(f_T)^* f_T X_{ij} = s_i^2 s_j^2 X_{ij}.$$

Hence $\mathcal{X}_B$ forms an orthonormal basis of eigenvectors for $\mathcal{L}(V)$, which implies that the eigenvalues of $f_T^* f_T$ are $s_i^2 s_j^2$ for $1 \leq i, j \leq n$, which implies that the singular values of f_T are $s_i s_j$ for $1 \leq i, j \leq n$.

3. (20 points) Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries all satisfy $a_{ij} \geq 0$ and whose diagonal entries are all positive. Let $D = (V, E)$ be the directed graph with vertex set

$$V = \{1, 2, \dots, n\}$$

and with edge set

$$E = \{i \to j : a_{ij} > 0\}.$$

In other words, there is a directed edge from vertex i to vertex j if and only if $a_{ij} > 0$, and the edge $i \to i$ belongs to E for $1 \leq i \leq n$. Prove that if

D has no directed cycles of even length then $\det A > 0$. Hint: Let $\sigma \in \mathcal{S}_n$. Prove that $\operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0$ if and only if σ factors into disjoint odd cycles and prove that every odd cycle has even sign.

Solution: Let σ be a permutation and suppose

$$a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0.$$

Let $\sigma_1, \ldots, \sigma_k$ be the cycles in the disjoint cycle-decomposition of σ. Let $\gamma = \sigma_i$ be the i^{th} cycle and suppose $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_j)$. Then

$$a_{\gamma_1,\gamma_2} a_{\gamma_2,\gamma_3} \cdots a_{\gamma_{j-1}\gamma_j} a_{\gamma_j\gamma_1}$$

is a divisor of

$$a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

therefore

$$a_{\gamma_1,\gamma_2} a_{\gamma_2,\gamma_3} \cdots a_{\gamma_{j-1}\gamma_j} a_{\gamma_j\gamma_1} \neq 0.$$

This implies

$$a_{\gamma_1,\gamma_2} \neq 0,\ a_{\gamma_2,\gamma_3} \neq 0,\ \ \ldots, a_{\gamma_{j-1}\gamma_j} \neq 0,\ a_{\gamma_j\gamma_1} \neq 0.$$

This implies

$$\gamma_1 \to \gamma_2 \in E,\ \gamma_2 \to \gamma_3 \in E,\ \ \ldots, \gamma_{j-1} \to \gamma_j \in E,\ \gamma_j \to \gamma_1 \in E.$$

This implies

$$\gamma_1 \to \gamma_2 \to \cdots \to \gamma_j \to \gamma_1$$

is a cycle in D. Since D has no even cycles in it, j must be an odd number. Therefore γ is an odd cycle. We have just shown that σ factors into odd cycles. Since a j-cycle can be factored into $j-1$ 2-cycles, the sign of a j-cycle is $(-1)^{j-1}$. In particular, the sign of an odd cycle is 1. Therefore σ factors into cycles each with sign equal to 1, which implies that the sign of σ is equal to one. To summarize, whenever σ is a permutation and contributes a non-zero term to the determinant, that term is a positive number since both the sign and the matrix entries in that term are all positive. Hence the determinant must be a sum of positive and zero contributions. Since the identity permutation contributes the term

$$a_{11} a_{22} \cdots a_{nn}$$

and we are assuming that the diagonal entries are all positive, the determinant has at least one non-zero contribution. Therefore the determinant is a positive number.