Exam 3 Solutions Math 447/547 Spring 2013

Due Thursday, April 11

Show all work. Be thorough!

Review of Inner-Product Spaces:

Let V be a vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$, finite or infinite-dimensional. An inner product on V is a function $\langle \bullet, \bullet \rangle : V \times V \to F$ which satisfies the following axioms:

1. **Positive-Definiteness:** $\langle v, v \rangle \geq 0$ for all $v \in V$, and $\langle v, v \rangle = 0$ if and only if $v = 0_V$.

2. **Multilinearity:** $\langle v + v', w \rangle = \langle v, w \rangle + \langle v', w \rangle$ and $\langle av, w \rangle = a\langle v, w \rangle$ for all $v, v', w \in V$ and $a \in W$.

3. **Conjugate Symmetry:** $\langle w, v \rangle = \overline{\langle v, w \rangle}$ for all $v, w \in V$.

Inner-Product Space: A real or complex vector space V equipped with an inner-product.

Exam Questions:

1. (30 points) Let $V = F$ where F where $F \in \{\mathbb{R}, \mathbb{C}\}$. Then V is a one-dimensional vector space over F. Let $\langle \bullet, \bullet \rangle$ be an inner product on V. True or false: $\langle 1, 1 \rangle = 1$. If true, prove it carefully using the three axioms of inner products: Positive-Definiteness, Multilinearity, Conjugate Symmetry. If false, let

$$X = \{c \in F : \text{ there exists an inner product on } V \text{ with } \langle 1, 1 \rangle = c\}$$

and carefully identify all the elements of X. Note that to prove $c \in X$ you must construct an inner product satisfying $\langle 1, 1 \rangle = c$ and prove that your inner product satisfies the three axioms.

Solution: False. In fact, $X = (0, \infty)$. Reason: Conjugate symmetry requires $\langle 1, 1 \rangle \in \mathbb{R}$. Positive-definiteness rules out $\langle 1, 1 \rangle \leq 0$. Therefore $X \subseteq (0, \infty)$. Now let $c \in (0, \infty)$. Define $\langle x, y \rangle = cx\overline{y}$. Then $\langle x, x \rangle = c|x|^2 \geq 0$, and this equals 0 if and only if $x = 0$. Hence positive-definiteness is met. Also, $\langle x+y, z \rangle = c(x+y)\overline{z} = cx\overline{z} + cy\overline{z} = \langle x, z \rangle + \langle y, z \rangle$ and $\langle ax, y \rangle = cax\overline{y} = acx\overline{y} = a\langle x, y \rangle$, hence multilineary is met. Finally, $\langle y, x \rangle = cy\overline{x} = \overline{cx\overline{y}} = \overline{\langle x, y \rangle}$, hence conjugate symmetry is met. Hence $c \in X$. Therefore $(0, \infty) \subseteq X$. Therefore $X = (0, \infty)$.

2. (30 points) Let V be a real vector space with inner product $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$. Let $W = V \times V$ with vector addition defined by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2),$$

scalar multiplication defined by

$$(a + bi)(v, w) = (av - bw, aw + bv),$$

and inner product $\langle \bullet, \bullet \rangle' : W \times W \to \mathbb{C}$ defined by

$$\langle (v_1, w_1), (v_2, w_2) \rangle' = (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle) + i(-\langle v_1, w_2 \rangle + \langle w_1, v_2 \rangle).$$

(a) Prove that W satisfies the axioms of a complex vector space.

(b) Prove that W has dimension n over $\mathbb{C}$, assuming that V has dimension n over $\mathbb{R}$.

(c) Prove that the inner product defined on W satisfies the three axioms of inner products over $\mathbb{C}$ (Positive-Definiteness, Multilinearity, Conjugate Symmetry), assuming that the inner product on V does over $\mathbb{R}$.

Solution: (a) One can check that W is an abelian group with additive identity $0_W = (0_V, 0_V)$ and additive inverse $-(v, w) = (-v, -w)$. We must also check $(rs) \cdot v = r \cdot (s \cdot v)$, $1 \cdot v = v$, $r \cdot (v + w) = (r \cdot v) + (s \cdot v)$ for all $r, s \in \mathbb{C}$ and $v, w \in W$. Write $r = r_1 + r_2 i$, $s = s_1 + s_2 i$, $v = (v_1, v_2)$, $w = (w_1, w_2)$.

$(rs) \cdot v = r \cdot (s \cdot v)$:

$$(rs) \cdot v = ((r_1 s_1 - r_2 s_2) + (r_1 s_2 + r_2 s_1) i)(v_1, v_2) =$$

$$((r_1 s_1 - r_2 s_2) v_1 - (r_1 s_2 + r_2 s_1) v_2), (r_1 s_1 - r_2 s_2) v_2 + (r_1 s_2 + r_2 s_1) v_1),$$

$$r \cdot (s \cdot v) = (r_1 + r_2 i)(s_1 v_1 - s_2 v_2, s_1 v_2 + s_2 v_1) =$$

$$(r_1(s_1 v_1 - s_2 v_2) - r_2(s_1 v_2 + s_2 v_1), r_1(s_1 v_2 + s_2 v_1) + r_2(s_1 v_1 - s_2 v_2).$$

These two expressions are the same.

$1 \cdot v = v$:

$$1 \cdot v = (1 + 0i)(v_1, v_2) = (1v_1 - 0v_2, 1v_2 + 0v_1) = (v_1, v_2) = v.$$

$r \cdot (v + w) = (r \cdot v) + (s \cdot v)$:

$$(r_1+r_2i)(v_1+w_1, v_2+w_2) = (r_1(v_1+w_1)-r_2(v_2+w_i), r_1(v_2+w_2)+r_2(v_1+w_1)) =$$

$$(r_1v_1 - r_2v_2, r_1v_2 + r_2v_1) + (r_1w_1 - r_2w_2, r_1w_2 + r_2w_1) =$$
$$(r_1 + r_2i)(v_1, v_2) + (r_1 + r_2i)(w_1, w_2).$$

(b) Let $\{v_1, \dots, v_n\}$ be a basis for V. We claim that $\{(v_1, 0), \dots, (v_n, 0)\}$ is a basis for W. We must verify that these vectors span W over $\mathbb{C}$ and are linearly independent over $\mathbb{C}$.

Span: Let $(v, w) \in W$ be given Write $v = \sum_{k=1}^n a_k v_k$ and $w = \sum_{k=1}^n b_k v_k$ where $a_k, b_k \in \mathbb{R}$ for each k. Then

$$(v, 0) = (\sum_{k=1}^n a_k v_k, 0) = \sum_{k=1}^n a_k (v_k, 0)$$

and

$$(w, 0) = (\sum_{k=1}^n b_k v_k, 0) = \sum_{k=1}^n b_k (v_k, 0),$$

therefore

$$(v, w) = (v, 0) + (0, w) = (v, 0) + i(w, 0) =$$
$$\sum_{k=1}^n a_k (v_k, 0) + i \sum_{k=1}^n b_k (v_k, 0) = \sum_{k=1}^n (a_k + b_k i)(v_k, 0).$$

Linear independence: Suppose

$$\sum_{k=1}^n (a_k + b_k i)(v_k, 0) = (0, 0).$$

Using the formula we derived in the previous paragraph, we then have

$$(\sum_{k=1}^{n} a_k v_k, \sum_{k=1}^{n} b_k v_k) = (0,0).$$

Therefore

$$\sum_{k=1}^{n} a_k v_k = \sum_{k=1}^{n} b_k v_k = 0.$$

By linear independence of $v_1, \ldots, v_n$ we have

$$a_1 = \cdots = a_n = b_1 = \cdots = b_n = 0.$$

Therefore

$$a_1 + b_1 i = \cdots = a_n + b_n i = 0.$$

(c) Positive-definiteness: We have

$$\langle (v,w),(v,w)\rangle = (\langle v,v\rangle + \langle w,w\rangle) + i(-\langle v,w\rangle + \langle w,v\rangle) = \langle v,v\rangle + \langle w,w\rangle \geq 0.$$

Suppose $\langle (v,w),(v,w)\rangle = 0$. Then

$$\langle v,v\rangle + \langle w,w\rangle = 0.$$

Since $\langle v,v\rangle \geq 0$ and $\langle w,w\rangle \geq 0$, we must have

$$\langle v,v\rangle = \langle w,w\rangle = 0.$$

This implies $v = w = 0_V$. Therefore

$$(v,w) = (0_V, 0_V) = 0_W.$$

Multilinearity: Let $x = (v,w)$, $x' = (v',w')$, $y = (v'',w'')$, $a = r_1 + r_2 i$. Then

$$\langle x + x', y\rangle = \langle (v+v', w+w'), (v'', w'')\rangle =$$

$$(\langle v+v', v''\rangle + \langle w+w', w''\rangle) + i(-\langle v+v', w''\rangle + \langle w+w', v''\rangle) =$$

$$(\langle v,v''\rangle + \langle v',v''\rangle + \langle w,w''\rangle + \langle w',w''\rangle) + i(-\langle v,w''\rangle - \langle v',w''\rangle + \langle w,v''\rangle + \langle w',v''\rangle) =$$

$$[(\langle v, v''\rangle + \langle w, w''\rangle) + i(-\langle v, w''\rangle + \langle w, v''\rangle)]+$$
$$[(\langle v', v''\rangle + \langle w', w''\rangle) + i(-\langle v', w''\rangle + \langle w', v''\rangle)] =$$
$$\langle (v, w), (v'', w'')\rangle + \langle (v', w'), (v'', w'')\rangle = \langle x, y\rangle + \langle x', y'\rangle.$$

Conjugate symmetry: Let $x = (v_1, w_1)$, $y = (v_2, w_2)$. Then

$$\langle x, y\rangle = (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle) + i(-\langle v_1, w_2\rangle + \langle v_2, w_1\rangle) =$$
$$(\langle v_2, v_1\rangle + \langle w_2, w_1\rangle) + i(-\langle w_2, v_1\rangle + \langle w_1, v_2\rangle) =$$
$$(\langle v_2, v_1\rangle + \langle w_2, w_1\rangle) - i(-\langle w_1, v_2\rangle + \langle w_2, v_1\rangle) =$$
$$\overline{(\langle v_2, v_1\rangle + \langle w_2, w_1\rangle) + i(-\langle w_1, v_2\rangle + \langle w_2, v_1\rangle)} =$$
$$\overline{\langle y, x\rangle}.$$

3. (40 points) Suppose a lake is stocked with 1000 fish and the population of fish is observed to be 2000 after 1 year, 4200 after 2 years, and 8300 after 3 years. You are asked to make a prediction of the fish population after 5 years. One approach is to assume an exponential model of population growth, $P(t) = Ae^{kt}$ where t is years and $P(t)$ is population after t years. For example, you could choose $A = 1000$ and $k = \ln 2$, but this does not fit the data exactly, and no choice of A and k will. Your task is to choose A and k appropriately to find the best fit in some well-defined sense, then compute $P(5)$. Using properties of inner-product spaces and orthogonal projection, provide a reasonable criterion for choosing A and k, then compute $P(5)$. The following elements must appear in your solution: (a) define a vector space V, (b) define the inner product on V, (c) define a subspace U in terms of the data provided, (d) express your criterion for choosing A and k in terms of orthogonal projection onto U, (e) cite the appropriate theorem that guarantees that your criterion is met, (f) compute A, k, and $P(5)$. It would be interesting to plot the data and the curve $y = Ae^{kt}$ on the same coordinate system, but this is not required.

Hint: $y = Ae^{kt}$ if and only if $\ln y = \ln A + kt$.

Solution: Let

$$(t_1, t_2, t_3, t_4) = (0, 1, 2, 3)$$

and

$$(L_1, L_2, L_3, L_4) = (\ln 1000, \ln 2000, \ln 4200, \ln 8300).$$

We want to choose A and k to minimize

$$\sqrt{\sum_{i=1}^{4} (L_i - \ln A - kt_i)^2}.$$

In other words, setting

$$L = (L_1, L_2, L_3, L_4) = (6.90776, 7.6009, 8.34284, 9.02401),$$

$$v_1 = (1, 1, 1, 1),$$
$$v_2 = (t_1, t_2, t_3, t_4) = (0, 1, 2, 3),$$

we wish to find minimize

$$||L - (\ln A \cdot v_1 + k \cdot v_2)||.$$

Setting $V = \mathbb{R}^4$, using the dot product as the inner product, and setting $U = \text{span}(v_1, v_2)$, we know that the unique vector in U which minimizes $||L - u||$ is $u = PL$ where $P : V \to U$ is orthogonal projection onto U. So we must find an orthonormal basis $\{u_1, u_2\}$ for U, set $u = \langle L, u_1 \rangle u_1 + \langle L, u_2 \rangle u_2$, then solve the equation

$$u = \ln A \cdot v_1 + k \cdot v_2$$

for A and k. Mathematica yields

$$u_1 = (0.5, 0.5, 0.5, 0.5)$$

$$u_2 = (-0.67082, -0.223607, 0.223607, 0.67082)$$
$$u = (6.90527, 7.61434, 8.32341, 9.03248)$$
$$\ln A = 6.90527$$
$$k = 0.70907$$

hence

$$P(t) = e^{6.90527} e^{0.70907t} = 997.519 e^{0.7907t},$$
$$P(5) = 34565.8.$$

So we predict that there will be 34566 fish in the lake in year 5. See the Mathematica notebook online.