Exam 2 Solutions Math 447/547 Spring 2013

Due Thursday, March 7

Show all work. Be thorough!

1. Let $V = P_2(\mathbb{R})$, $W = P_3(\mathbb{R})$, and let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = 3x + 1$. Define $T \in \mathcal{L}(V, W)$ via $T(f) = hf$ (product of functions). Since T is injective and W is finite-dimensional, by Problem 14 of Chapter 3 we know that there exists $S \in \mathcal{L}(W, V)$ such that $ST = I$. Let

$$X = \{S \in \mathcal{L}(W, V) : ST = \lambda I \text{ for some } \lambda \in \mathbb{R}\}.$$

(a) Prove that X is a subspace of $\mathcal{L}(W, V)$ and (b) compute its dimension.

Hint: Represent all linear maps by suitable matrix representations. You should be able to derive a system of equations for the 12 entries of S. Solve this system, use the solutions to find a basis for $\{M(S) : S \in X\}$, and use this to compute the dimension of X.

Solution: (a) Let $S_1, S_2 \in X$ and $a_1, a_2 \in \mathbb{R}$. Then there exist real numbers λ_1, λ_2 such that $S_1T = \lambda_1 I$ and $S_2T = \lambda_2 I$, therefore $(a_1S_1 + a_2S_2)T = a_1S_1T + a_2S_2T = a_1\lambda_1 I + a_2\lambda_2 I = (a_1\lambda_1 + a_2\lambda_2)I$, therefore $a_1S_1 + a_2S_2 \in X$. Hence X is a subspace of $\mathcal{L}(W, V)$.

(b) We will predict the dimension of X first, given our understanding of how S is formed: Expand the linearly independent vectors $T1$, Tx, Tx^2 to a basis T_1, Tx, Tx^2, w for W, then define S on this basis via $S(T1) = 1$, $S(Tx) = x$, $S(Tx^2) = x^2$, $S(w) = v$ where $v \in V$ is arbitrary. Given that the two choices w and v have to be made, I predict that $\dim X = 1 + 3 = 4$. Let's do this carefully however.

To compute $\dim X$, observe that $T1 = 3x + 1$, $Tx = 3x^2 + x$, and $Tx^2 = 3x^3 + x^2$, therefore the matrix representation of T with respect to the basis $\{1, x, x^2\}$ for V and the basis $\{1, x, x^2, x^3\}$ for W is

$$M(T) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $S \in X$. Its matrix representation with respect to these two bases must be of the form

$$M(S) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Assuming $ST = \lambda I$, the matrix representation of λI with respect to the basis for V is

$$M(\lambda I) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Hence

$$M(S)M(T) - M(\lambda I) = M(ST - \lambda I) = M(0) = 0.$$

This yields

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix equation implies a system of 9 equations satisfied by the 13 variables $a_{11}, \ldots, a_{34}, \lambda$. Writing these out and then expressing the result in matrix form yields

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{31} \\ a_{32} \\ a_{33} \\ a_{34} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The system of equations this implies can be represented by the augmented matrix

$$\left[\begin{array}{ccccccccccccc|c} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & -1 & 0 \end{array}\right].$$

Since this matrix is already in row-reduced form, and there are 9 leading 1s and 4 slack variables $(a_{14}, a_{24}, a_{34}, \lambda)$, in principle each of the other variables can be expressed in terms of these, which leads to another prediction that the dimension is 4. But we need to find a basis in order to accurately compute the dimension.

Solving the system

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right]$$

for the variables $a_{11}, a_{12}, a_{13}, a_{14}, \lambda$ yields

$$(a_{11}, a_{12}, a_{13}) = (-27a_{14} + \lambda, 9a_{14}, -3a_{14}).$$

Solving the system

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right]$$

for the variables $a_{21}, a_{22}, a_{23}, a_{24}, \lambda$ yields

$$(a_{21}, a_{22}, a_{23}) = (-27a_{24} - 3\lambda, 9a_{24} + \lambda, -3a_{24}).$$

Solving the system

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \end{array}\right]$$

for the variables $a_{31}, a_{32}, a_{33}, a_{34}, \lambda$ yields

$$(a_{31}, a_{32}, a_{33}) = (-27a_{34} + 9\lambda, 9a_{34} - 3\lambda, -3a_{34} + \lambda).$$

So we can write

$$M(S) = \begin{bmatrix} -27a_{14} + \lambda & 9a_{14} & -3a_{14} & a_{14} \\ -27a_{24} - 3\lambda & 9a_{24} + \lambda & -3a_{24} & a_{24} \\ -27a_{34} + 9\lambda & 9a_{34} - 3\lambda & -3a_{34} + \lambda & a_{34} \end{bmatrix} =$$

$$a_{14}A + a_{24}B + a_{34}C + \lambda D$$

where

$$A = \begin{bmatrix} -27 & 9 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -27 & 9 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -27 & 9 & -3 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 9 & -3 & 1 & 0 \end{bmatrix}.$$

The values of $a_{14}, a_{24}, a_{34}, \lambda$ can be chosen arbitrarily. Hence

$$\{M(S) : S \in X\} = \operatorname{span}\{A, B, C, D\}.$$

Since A, B, C, D can easily be seen to be linearly independent matrices,

$$\dim\{M(S) : S \in X\} = 4.$$

By the vector space isomorphism between $\mathcal{L}(W, V)$ and $M_{3,4}(\mathbb{R})$, $\dim(X) = 4$.

Note that A, B, and C represent linear operators N such that $NT = 0$, and D represents a linear operator S_0 such that $S_0T = I$. Hence the typical solution to $ST = \lambda I$ is $S = N + \lambda S_0$ where $N = pA + qB + rC$ for some $p, q, r \in \mathbb{R}$.

2. Let $V = P(\mathbb{Z}_5)$ and let $h : \mathbb{Z}_5 \to \mathbb{Z}_5$ be defined by $h(x) = 3x + 1$. Define $T \in \mathcal{L}(V)$ via $T(f) = f \circ h$ (composition of functions). (a) Find an upper-triangular matrix representation for T, (b) use it to identify the eigenvalues of λ, then (c) find a basis for each eigenspace $E_\lambda = \{v \in V : Tv = \lambda v\}$. Do not use determinants.

Hint: We proved in class that $P(\mathbb{Z}_5)$ has basis $\{1, x, x^2, x^3, x^4\}$.

Solution: (a) We have $T1 = 1$, $Tx = 3x+1$, $Tx^2 = (3x+1)^2 = 4x^2+x+1$, $Tx^3 = (3x+1)^3 = 2x^3+2x^2+4x+1$, $Tx^4 = (3x+1)^4 = x^4+3x^3+4x^2+2x+1$, therefore with respect to the basis $\{1, x, x^2, x^3, x^4\}$ we have

$$M(T) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The eigenvalues appear on the diagonal of $M(T)$. They are 1, 2, 3, 4.

(c) Let $v \in E_\lambda$. Then $M(T)M(v) = M(\lambda v)$, therefore $M(T - \lambda I)M(v) = 0$. We can use this relationship to compute $M(v)$, then express v in terms of the basis.

E_1 basis: $\{1, x^4 + 2x^3 + 4x^2 + 3x\} = \{1, (x+3)^4 - 1\}$

E_2 basis: $\{x^3 + 4x^2 + 2x + 2\} = \{(x+3)^3\}$

E_3 basis: $\{x + 3\}$

E_4 basis: $\{x^2 + x + 4\} = \{(x+3)^2\}$

Note that, with with respect to the basis $\{1, x+3, (x+3)^2, (x+3)^3, (x+3)^4\}$, T has matrix representation

$$M(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In other words,

$$M(T) = \begin{bmatrix} 3^0 & 0 & 0 & 0 & 0 \\ 0 & 3^1 & 0 & 0 & 0 \\ 0 & 0 & 3^2 & 0 & 0 \\ 0 & 0 & 0 & 3^3 & 0 \\ 0 & 0 & 0 & 0 & 3^4 \end{bmatrix}.$$