Exam 1 Solutions Math 447/547 Spring 2013

Due Thursday, February 7

Show all work. Be thorough!

1. Let V be the real vector space of functions from $(0, 1)$ to $\mathbb{R}$. Let W be the set of infinitely-differentiable functions in V. Let W^+ be the set of positive-valued functions in W. We will denote by f' the derivative of the function f.

(a) Prove that W is a subspace of V.

(b) Let f and g be vectors in W^+. Prove that f and g are linearly independent if and only if $fg' - f'g \neq 0_V$.

Hints: for a positive valued differentiable function h, $(\ln h)' = \frac{h'}{h}$; and $h' = k'$ implies $h = k + c$ for a constant function c.

Solution: (a) It suffices to prove that if f and g are infinitely differentiable and a and b are constants, then $af + bg$ is also infinitely differentiable. This follows from the fact that

$$(af + bg)^{(k)} = af^{(k)} + bg^{(k)},$$

which demostrates that deriviatives of $af + bg$ of all orders exist.

(b) We will actually show that f and g are linearly dependent if and only if $fg' - f'g = 0_V$, which is logically the same statement. We have $fg' - f'g = 0_V$ iff $(f'/f) = (g'/g)$ iff $(\ln f)' = (\ln g)'$ iff $\ln f = \ln g + c$ for some constant c iff $f = e^c g$ for some c iff f and g are dependent. Alternatively, f and g dependent iff $f = cg$ for some constant $c > 0$ iff $f/g = c$ for some $c > 0$ iff $(f/g)' = 0_V$ iff $(f'g - fg')/g^2 = 0_V$ iff $f'g - fg' = 0_V$.

2. Let $V = \mathbb{R}^\infty$, which we know to be an infinite-dimensional real vector space. Let

$$W = \{(a_0, a_1, a_2, \dots) \in V : \text{ for all } n \geq 2,\ a_n = 4a_{n-1} - 4a_{n-2}\}.$$

For any two real numbers p and q, the vector

$$(p, q, 4q - 4p, 12q - 16p, \cdots)$$

belongs to W.

(a) Prove that W is a subspace of V.

(b) Prove that W is finite-dimensional and find a basis for it.

Solution: (a) It suffices to show that $(a_0, a_1, \dots) \in W$ and $(b_0, b_1, \dots) \in W$ implies $p(a_0, a_1, \dots) + q(b_0, b_1, \dots) \in W$ for all $p, q \in \mathbb{R}$. Assuming $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geq 2$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for all $n \geq 2$, it follows after a spot of algebra that $c_n = 4c_{n-1} - 4c_{n-2}$ for all $n \geq 2$, where $c_n = pa_n + qb_n$. This is the n^{th} entry of $p(a_0, a_1, \dots) + q(b_0, b_1, \dots)$, hence the latter belongs to W.

(b) Let $e_0 = (1, 0, -4, -16, \dots) \in W$ and $e_1 = (0, 1, 4, 12, \dots) \in W$. They are linearly independent:

$$pe_0 + qe_1 = (0, 0, \dots)$$

implies

$$(p, q, 4q - 4p, 12q - 16p, \cdots) = (0, 0, \dots)$$

implies

$$p = q = 0.$$

They also span W, because if $(a_0, a_1, \dots) \in W$ then $a_0e_0 + a_1e_1 \in W$ as well, and since a sequence is completely determined by its first two terms by definition, it must be the case that

$$(a_0, a_1, \dots) = a_0e_0 + a_1e_1$$

since they both start with the same two terms and belong to W.