Final Exam Solutions Math 122 Fall 2012 Name:

Work must be shown to receive credit.

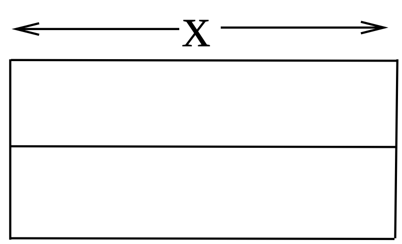
1. Let $f(x) = 9x - x^3$. (a) Make a sign chart for $f(x)$. (b) Find all local maximum and minimum values of $f(x)$. (c) Find all points of inflection on the curve $y = f(x)$.

Solution: (a) $f'(x) = 9 - 3x^2$ has roots $\pm\sqrt{3}$. $f''(x) = -6x$ has root 0. Sign chart is

x	$\cdots$	$-\sqrt{3}$	$\cdots$	0	$\cdots$	$\sqrt{3}$	$\cdots$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$

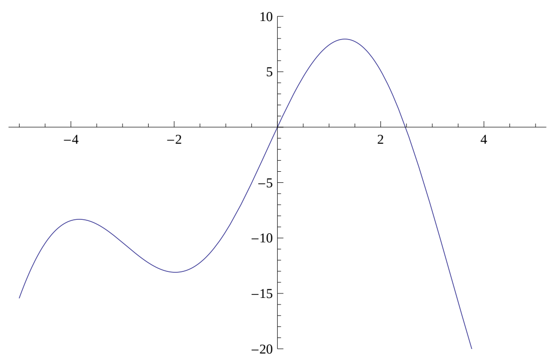
Local max value $f(\sqrt{3}) = 6\sqrt{3}$, local min value $f(-\sqrt{3}) = -6\sqrt{3}$, point of inflection at $(0, f(0)) = (0, 0)$.

2. A rectangular pen with a middle partition as in the figure below is to be constructed using 50 linear feet of fencing material. Find the maximum possible area enclosed by the pen by maximizing the function with input x, output area. Use calculus to solve the problem.



Solution: $3x$ feet are used for the horizontal sides, leaving $50-3x$ feet to divide evenly between the two vertical slides. This implies an area of $f(x) = x\left(\frac{50-3x}{2}\right) = \frac{1}{2}(50x - 3x^2)$. Given that $f'(x) = \frac{1}{2}(50 - 6x)$, a critical value occurs at $x = 50/6 = 25/3$. Given that $f''(x) = -3$, the second derivative test implies that $f(x)$ attains a maximum value at $x = 25/3$. The maximum area is therefore $f(25/3) = 625/6 = 104\frac{1}{6}$ square feet.

3. Let $f(x) = 10\sin x - x^2$. A sketch of this graph is provided below. The maximum output value of the function is somewhere between 5 and 10 and can be obtained using an input value of x somewhere between 0 and 2, where $f'(x) = 0$. Use Newton's Method to find the critical value of $f(x)$ and the maximum output value of $f(x)$, starting with $x_0 = 1$ and computing x_1, x_2, x_3 and $f(x_3)$.



Solution: The critical value occurs where $f'(x) = 0$, i.e. $10\cos x - 2x = 0$. We solve this equation using Newton's Method applied to $F(x) = 10\cos x - 2x$. Since $F'(x) = -10\sin x - 2$, Newton's Method yields

$$x_{n+1} = x_n - \frac{10\cos x - 2x}{-10\sin x - 2}.$$

Using $x_0 = 1$ we obtain $x_1 = 1.32675$, $x_2 = 1.30648$, $x_3 = 1.30644$. The approximate critical value is 1.30644. $f(1.30644) = 7.94582$, so the minimum value of the function is approximately 7.94582.

4. An oil tanker begins leaking oil at time $t = 0$ and continues to leaks oil at a rate of $1000 - t^3$ gallons/hour at time t hours for $0 \leq t \leq 10$. How much oil does the oil tanker leak during these 10 hours?

Solution: If $f(t)$ is the number of gallons lost at time t, then $f'(t) = 1000 - t^3$. This implies $f(t) = 1000t - t^4/4 + C$ for some C. Initial conditions imply $C = 0$, hence $f(t) = 1000t - t^4/4$. At time 10 we have lost $f(10) = 7500$ gallons of oil.

5. A car at rest (initial velocity 0 at time 0) starts accelerating at a constant rate of 10 feet/second^2. How many seconds does it take the car to travel 800 feet?

Solution: If the acceleration is 10 then the velocity is $10t$ and the distance traveled is $5t^2$ after t seconds. Solving the equation $5t^2 = 800$ we obtain $t = \sqrt{800/5} \approx 12.6491$ seconds.

6. Approximate the definite integral $\int_0^{10} x^3 \, dx$ using a Riemann Sum with $N = 5$ equally spaced subintervals and the midpoint rule to generate x_1^* through x_5^*.

Solution: The partition points are $x_i = 0, 2, 4, 6, 8, 10$ and the midpoint rule yields $x_i^* = 1, 3, 5, 7, 9$. The resulting Riemann Sum is

$$1^3(2) + 3^3(2) + 5^3(2) + 7^3(2) + 9^3(2) = 2450.$$

7. Find the exact value of $\int_1^2 \frac{1+x}{x^3}\, dx$. Note that the antiderivative is not $\frac{x+\frac{x^2}{2}}{\frac{x^4}{4}} + C$.

Solution: The integrand is $f(x) = \frac{1+x}{x^3} = x^{-3} + x^{-2}$. An antiderivative is $F(x) = \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} = -\frac{1}{2x^2} - \frac{1}{x}$. The definite integral is $F(2) - F(1) = \frac{7}{8}$.

8. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \sin x(1+\cos x)^{10}\, dx$ using u-substitution.

Solution: $u = 1 + \cos x$, $\frac{du}{dx} = -\sin x$, $du = -\sin x\ dx$, $-du = \sin x\ dx$, $u(0) = 2$, $u(\frac{\pi}{2}) = 1$,

$$\int_0^{\frac{\pi}{2}} \sin x(1+\cos x)^{10}\ dx = -\int_2^1 u^{10}\ du = -\frac{1}{11}u^{11}\Big|_2^1 = \frac{1}{11}(2^{11}-1) = \frac{2047}{11}.$$