Exam 3 Math 121 Spring 2013

Show all work.

1. Given that $\ln 1.0 = 0$, estimate the value of $\ln 0.9$ using linear approximation. Do not use a calculator.

Solution: Using $f(x) \approx f(a) + f'(a)(x - a)$ for x close to a, and using $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $a = 1.0$, $f(a) = 0$, $f'(a) = 1$, $x = 0.9$, we obtain

$$\ln 0.9 \approx 0 + 1(0.9 - 1.0) = -0.1.$$

2. Let $f(x) = (\ln x)^x$. Compute $f'(x)$. The answer is not $\ln(\ln x)(\ln x)^x$, $x(\ln x)^{x-1}$, etc.

Solution: We can write

$$f(x) = e^{\ln f(x)} = e^{\ln((\ln x)^x)} = e^{x\ln(\ln x)}.$$

Therefore

$$f'(x) = (x\ln(\ln x))'e^{x\ln(\ln x)} = (\ln(\ln x) + x\frac{\frac{1}{x}}{\ln x})e^{x\ln(\ln x)} =$$

$$(\ln(\ln x) + \frac{1}{\ln x})(\ln x)^x.$$

3. A lake is stocked with a certain number of fish. The fish population one year later is 1750, and two years later is 4500. Assuming the population is growing exponentially, how many fish was the lake stocked with originally? (In other words, $f(1) = 1750$, $f(2) = 4500$, $f(0) = ?$)

Solution: Set $f(t) = Ae^{kt}$. We want to find A. The information provided yields $Ae^k = 1750$ and $Ae^{2k} = 4500$. This yields

$$\frac{4500}{1750} = \frac{Ae^{2k}}{Ae^k} = e^k.$$

Substituting this into $Ae^k = 1750$ yields

$$A\frac{4500}{1750} = 1750.$$

Therefore

$$A = \frac{1750^2}{4500} = 680.556.$$

So there were approximately 681 fish in the lake initially.

4. Using L'Hopital's Rule appropriately, compute

$$\lim_{x\to 0} \frac{\ln(x+1)-x}{x^2}.$$

Solution: The limit is indeterminate of form $\frac{0}{0}$. One application of L'Hopital's Rule yields

$$\lim_{x\to 0} \frac{\frac{1}{x+1}-1}{2x}.$$

This is still indeterminate of form $\frac{0}{0}$. A second application of L'Hopital's Rule yields

$$\lim_{x\to 0} \frac{\frac{-1}{(x+1)^2}}{2} = \frac{-1}{2}.$$

5. Let $f(x) = \tan x - 2x$ on the interval $[0, \frac{\pi}{3}]$.

(a) Find the critical value of $f(x)$ in the interval. Give the exact value, not a calculator approximation.

(b) Find the absolute minimum output value of $f(x)$ on the interval. Give the exact value, not a calculator approximation.

Solution: We have $f'(x) = \sec^2 x - 2$. This is equal to zero when $\sec^2 x = 2$, $\sec x = \sqrt{2}$, $\cos x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $x = \frac{\pi}{4}$. To find the absolute minimum output value of $f(x)$ we need to compare $f(0)$, $f(\frac{\pi}{4})$, $f(\frac{\pi}{3})$. We have

$$f(0) = \tan 0 - 0 = 0$$

$$f(\frac{\pi}{4}) = \tan\frac{\pi}{4} - \frac{\pi}{2} = 1 - \frac{\pi}{2} \approx -0.570796,$$

$$f(\frac{\pi}{3}) = \tan\frac{\pi}{3} - \frac{2\pi}{3} \approx -0.362344.$$

So the absolute minimum output value is $f(\frac{\pi}{4}) = 1 - \frac{\pi}{2}$.