Exam 2 Math 121 Spring 2013 Name:

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1. Compute the derivative of $f(x) = x^3$ using the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Solution:

$$f'(x) = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} =$$

$$\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$$

as $h \to 0$.

2. Compute the derivative of $\frac{\cos x}{1+\sin x}$ using the quotient rule.

Solution:

$$\frac{-\sin x \cdot (1+\sin x) - \cos x \cdot (0+\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} =$$

$$\frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x}.$$

3. Compute the derivative of $\sqrt[4]{x^2+5x}$ using the chain rule.

Solution: $\sqrt[4]{x^2+5x} = f(g(x))$ where $f(x) = x^{\frac{1}{4}}$, $g(x) = x^2+5x$. Given that $f'(x) = \frac{1}{4}x^{-3/4}$ and $g'(x) = 2x+5$, the derivative is

$$f'(g(x))g'(x) = \frac{1}{4}(x^2+5x)^{-3/4}(2x+5).$$

4. Let C be the curve defined by $xy^2 + yx^2 = 2$.

(a) Using implicit differentiation, find a formula for $\frac{dy}{dx}$.

(b) Find the equation of the line tangent to the curve C at the point $(1, 1)$.

Solution: (a) Differentiating the equation of the curve with respect to x, we obtain

$$(1 \cdot y^2 + x \cdot 2y\frac{dy}{dx}) + (\frac{dy}{dx} \cdot x^2 + y \cdot 2x) = 0.$$

Rearranging,

$$\frac{dy}{dx}(2xy + x^2) = -y^2 - 2yx.$$

Dividing, we obtain

$$\frac{dy}{dx} = \frac{-y^2 - 2yx}{2xy + x^2}.$$

(b) At $(1, 1)$, $\frac{dy}{dx} = \frac{-3}{3} = -1$. Hence the tangent slope is -1. Point-slope formula for tangent line yields

$$y - 1 = -1(x - 1)$$

or

$$y = -x + 2.$$

5. Consider a spherical balloon filled with helium gas, floating up and growing larger over time as the atmosphere thins. We can measure both pressure P in the balloon (in units of psi) and the volume V of the balloon (in units of cubic inches) over time t (in units of minutes). Boyle's Law predicts that P and V are related by the equation $PV = 200$. If the pressure in the balloon is changing at a rate of -5 psi/min at the exact moment that the volume of the balloon is 100 in^3, find how fast the volume is increasing in units of in^3/min at this time.

Solution: Differentiating $PV = 200$ with respect to t we obtain

$$\frac{dP}{dt}V + P\frac{dV}{dt} = 0.$$

Substituting $\frac{dP}{dt} = -5$ and $V = 100$, we obtain

$$-500 + P\frac{dV}{dt} = 0.$$

Give that $PV = 200$ and $V = 100$, we must have $P = 2$ at this moment. Hence

$$-500 + 2\frac{dV}{dt} = 0.$$

Therefore

$$\frac{dV}{dt} = 250.$$