Triangularizing a Real Symmetric Matrix

We know that if A is a real symmetric matrix then there is an invertible matrix C and a diagonal matrix D such that $C^{-1}AC = D$. This implies that $\mathbb{R}^n$ has a basis of eigenvectors of A. Moreover, the number of basis eigenvectors corresponding to an eigenvalue λ is equal to the number of times λ occurs as a root of the characteristic polynomial. Find a basis for each eigenspace. We must argue that the union of each eigenspace basis is linearly independent. Since there are the right number of vectors if that is the case, they form a basis.

We will show that if u and v are eigenvectors corresponding to distinct eigenvectors then $u^T v = 0$. Hence if we find an orthonormal basis for each eigenspace, then the union of them will be orthonormal and so will be linearly independent. This will imply also that $C^T = C^{-1}$.

Suppose $A^T = A$ and $Au = \alpha u$ and $Av = \beta v$ where $\alpha \neq \beta$. Then

$$\beta u^T v = u^T(\beta v) = u^T(Av) = (u^T A)v =$$

$$(u^T A^T)v = (Au)^T v = (\alpha u)^T v = \alpha(u^T v).$$

Since $\alpha \neq \beta$, this forces $u^T v = 0$. In other words, their dot product is 0.