Derivation of Jordan Normal Form

$n \times n$ Jordan block: $J_\lambda(n)$. A block matrix of these: $J_\lambda(n_1, n_2, \ldots, n_k)$.

Property of a linear transformation with matrix representation $J_0(n)$: If basis is

$$\{v_1, v_2, \ldots, v_n\}$$

then we have

$$f(v_n) = v_{n-1}, f^2(v_n) = v_{n-2}, \ldots, f^{n-1}(v_n) = v_1, f^n(v_n) = 0.$$

So the basis can be expressed in the form

$$\{f^{n-1}(u_1), f^{n-2}(u_2), \ldots, f^0(u_1)\}$$

where $u_1 = v_n$. More generally, a linear transformation with matrix representation $J_0(n_1, n_2, \ldots, n_k)$ has a basis of the form

$$\{f^{n_1-1}(u_1), \ldots, f^0(u_1), f^{n_2-1}(u_2), \ldots, f^0(u_2), \ldots, f^{n_k-1}(u_k), \ldots, f^0(u_k)\}$$

where

$$f^{n_1}(u_1) = f^{n_2}(u_2) = \cdots = f^{n_k}(u_k) = 0.$$

Clearly the transformation is nilpotent.

Lemma: If $f : V \to V$ is nilpotent linear map of a finite-dimensional vector space then V has a basis of the form

$$\{f^{n_1-1}(u_1), \ldots, f^0(u_1), f^{n_2-1}(u_2), \ldots, f^0(u_2), \ldots, f^{n_k-1}(u_k), \ldots, f^0(u_k)\}$$

where

$$f^{n_1}(u_1) = f^{n_2}(u_2) = \cdots = f^{n_k}(u_k) = 0.$$

Hence f has a matrix representation in the form $J_0(n_1, n_2, \ldots, n_k)$.

Proof: By induction on the dimension of V. Trivial in dimension 1. Assume true for dimension $\leq n$. Consider dimension $n+1$. Let $W = f(V)$. Since f is nilpotent, $W \neq V$. If $W = 0$ then the zero-matrix represents f. Otherwise, W has dimension smaller than the dimension of V and $f : W \to W$ is nilpotent. By the induction hypothesis, W has a basis of the form

$$\{f^{n_1-1}(v_1), \ldots, f^0(v_1), f^{n_2-1}(v_2), \ldots, f^0(v_2), \ldots, f^{n_k-1}(v_k), \ldots, f^0(v_k)\}$$

where

$$f^{n_1}(v_1) = f^{n_2}(v_2) = \cdots = f^{n_k}(v_k) = 0.$$

Find u_i so that $f(u_i) = v_i$ for each $i \leq k$. Expand the vectors $f^{n_1-1}(v_1), \ldots, f^{n_k-1}(v_k)$, which belong to the kernel of f, to a basis $f^{n_1-1}(v_1), \ldots, f^{n_k-1}(v_k), u_{k+1}, \ldots, u_p$.
We claim that the desired basis for V is

$$f^{n_1}(u_1), \ldots, f^0(u_1), f^{n_k}(u_k), \ldots, f^0(u_{k+1}), \ldots, f^0(u_p).$$

We must show that these vectors are linearly independent and that there are the right number of them.

They are linearly independent: Suppose

$$\sum_{i=1}^{k} \sum_{j=0}^{n_i} \alpha_{ij} f^j(u_i) + \sum_{i=k+1}^{p} \beta_i f^0(u_i) = 0_V.$$

Applying f we obtain

$$\sum_{i=1}^{k} \sum_{j=0}^{n_i} \alpha_{ij} f^j(v_i) = 0_V.$$

Therefore the $\alpha_{ij} = 0$ for $i < n_i$. This implies

$$\sum_{i=1}^{k} \alpha_{in_i} f^{n_i}(u_i) + \sum_{i=k+1}^{p} \beta_i f^0(u_i) = 0_V.$$

In other words,

$$\sum_{i=1}^{k} \alpha_{in_i} f^{n_i-1}(v_i) + \sum_{i=k+1}^{p} \beta_i f^0(u_i) = 0_V.$$

By linear independence, the remaining coefficients are equal to zero.

There are the right number of them: The dimension of W is $n_1 + \cdots + n_k$. Hence the rank of f is $n_1 + \cdots + n_k$. We also have that the nullity of f is p. By the Rank-Nullity theorem, the dimension of V is $n_1 + \cdots + n_k + p$. This is the number of linearly independent vectors we have produced.

Theorem: Every linear transformation $f : V \to V$ of a finite-dimensional vector space V over the complex numbers has a block-diagonal matrix representation where each diagonal block is of the form $J_\lambda(n_1, \ldots, n_k)$.

Proof:

First, get matrix into triangular form with eigenvalues along diagonals.

Second, change basis so that the matrix is in block diagonal form with constant diagonals.

Third, show each block A can be placed in Jordan Normal Form.

Proof of third step: Suppose the eigenvalue corresponding to A is λ. Then $A - \lambda I$ is the matrix representation of a nilpotent transformation. So the restriction of $f - \lambda e$ to the corresponding subspace is nilpotent. Hence $f - \lambda e$ has a matrix representation of the form $J_0(n_1, \ldots, n_k)$, and this implies that f has a matrix representation of the form $J_\lambda(n_1, \ldots, n_k)$.