Effective procedure for computing the Jordan Normal Form of nilpotent matrix

Given an $n \times n$ nilpotent matrix A, define the linear transformation

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

by

$$f(x) = Ax.$$

Then A represents f with respect to the standard basis. Define the subspaces

$$V_0 = \mathbb{R}^n,\ V_1 = cs(A),\ V_2 = cs(A^2),\ \ \ldots,\ V_n = cs(A^n).$$

Note that f maps V_k into V_k for each k: This is clear when $k = 0$. For $k \geq 1$, let $x \in V_k$ be given. Then we have $x = A^k y$ for some $y \in \mathbb{R}^n$, hence

$$f(x) = Ax = A^{k+1}y = A^k(Ay) \in cs(A^k) = V_k.$$

For each k define

$$f_k : V_k \to V_k$$

by

$$f_k(x) = Ax.$$

Then each f_k is a nilpotent transformation. We will find a Jordan normal basis for each V_k, working backwards from the largest value of k such that $A^k \neq 0$.

Find the largest value of k such that $A^k \neq 0$. We must have $A^{k+1} = 0$. We also have $V_k = cs(A^k) \neq \{0\}$ and $f_k(x) = f_k(A^k y) = A^{k+1}y = 0$ for all $x \in V_k$. Hence f_k is represented by the zero matrix with respect to any basis for V_k. The size of this basis is determined by the dimension of V_k. Since the zero matrix is Jordan normal of type $J_0(1, 1, \ldots)$, we have found a Jordan normal basis for V_k.

Induction Hypothesis: we have found a Jordan normal basis for V_j where $j \geq 1$. Assume it has the form

$$\{A^{n_1-1}v_1, \ldots, A^0 v_1, A^{n_2-1}v_2, \ldots, A^0 v_2, \ldots, A^{n_p-1}v_k, \ldots, A^0 v_k\}$$

where

$$A^{n_1}(v_1) = A^{n_2}v_2 = \cdots = A^{n_k}v_k = 0.$$

Now we find a Jordan normal basis for V_{j-1}. Find u_i so that $A(u_i) = v_i$ for each $i \leq k$. Expand the vectors $A^{n_1-1}(v_1), \ldots, A^{n_k-1}(v_k)$, which belong to the kernel of f_{j-1}, to a basis $A^{n_1-1}(v_1), \ldots, A^{n_k-1}(v_k), v_{k+1}, \ldots, v_p$ for $\ker(f_j)$. Then by a previous theorem,

$$A^{n_1}u_1, \ldots, A^0u_1, A^{n_k}u_k, \ldots, A^0u_k, A^0v_{k+1}, \ldots, A^0v_p$$

is a Jordan normal basis for V_{j-1}.

Example:

$$A = \begin{bmatrix} 0 & 1 & 7 & 19 & 54 & 61 \\ 0 & 0 & 0 & -7 & -29 & -41 \\ 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 11 & 66 \\ 0 & 0 & 0 & 0 & -7 & -42 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A Jordan normal basis for V_2 is

$$\left\{ \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

This belongs to the kernel of f_1. We must find other vectors we can add to this to create a basis for the kernel of f_1. In order to do this efficiently we must first find a matrix representation of f_1.

A basis for V_1 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 54 \\ -29 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The matrix representation of f_1 with respect to this basis is

$$\begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The kernel of this matrix is

$$\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This implies that a basis for the kernel of f_1 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We must find a vector v such that

$$\left\{ \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v \right\}$$

have the same span as

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We can choose

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We must also find u so that

$$Au = \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We choose

$$u = \begin{bmatrix} 54 \\ -29 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The Jordan normal basis we have found for V_1 is now

$$\{Au, u, v\} = \left\{ \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 54 \\ -29 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The vectors Au and v belong to the kernel of f_0. We will fill these out to a basis for the kernel of f_0. We already have a matrix representation of f_0: it is the matrix A with respect to the standard basis. The kernel of A has basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 98 \\ 0 \\ -19 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 14 \\ -19 \\ 6 \\ -1 \end{bmatrix} \right\}.$$

These vectors have the same span as the vectors

$$\left\{ \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 14 \\ -19 \\ 6 \\ -1 \end{bmatrix} \right\}.$$

So set

$$w = \begin{bmatrix} 0 \\ 0 \\ 14 \\ -19 \\ 6 \\ -1 \end{bmatrix}.$$

We must find also find vectors p and q so that $Ap = u$ and $Aq = v$. We can use $p = e_5$ and $q = e_2$. So a Jordan normal basis for V_0 should be

$$\{A^2p, Ap, p, Aq, q, w\} = \left\{ \begin{bmatrix} 11 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 54 \\ -29 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 14 \\ -19 \\ 6 \\ -1 \end{bmatrix} \right\}.$$

One can check that the matrix representation of f with respect to this basis is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = J_0(3,2,1).$$

Therefore

$$\begin{bmatrix} 11 & 54 & 0 & 1 & 0 & 0 \\ -7 & -29 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 14 \\ 0 & 1 & 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^{-1} A \begin{bmatrix} 11 & 54 & 0 & 1 & 0 & 0 \\ -7 & -29 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 14 \\ 0 & 1 & 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = J_0(3,2,1).$$

As a corollary, we get

$$\begin{bmatrix} 11 & 54 & 0 & 1 & 0 & 0 \\ -7 & -29 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 14 \\ 0 & 1 & 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda & 1 & 7 & 19 & 54 & 61 \\ 0 & \lambda & 0 & -7 & -29 & -41 \\ 0 & 0 & \lambda & 1 & 3 & -1 \\ 0 & 0 & 0 & \lambda & 1 & 6 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 11 & 54 & 0 & 1 & 0 & 0 \\ -7 & -29 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 14 \\ 0 & 1 & 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} =$$

$$J_\lambda(3,2,1).$$