Math 375

Selected Solutions to Homework 2 Problems

Section 1.4

3g. In order to prove that a graph is not planar it suffices to show that it has a subgraph which is not planar. So we don't need to find a Hamilton circuit through all the vertices, just through a nonplanar subgraph. Let's ignore the vertex a. Consider the cycle

$$b - e - h - c - d - i - f - g - b.$$

To find a K_{33} configuration we just need to find one (possibly interrupted) chord inside the cycle, forcing another (possibly interrupted) chord outside the cycle, preventing the drawing of any other (possibly interrupted) chord either inside or outside the cycle. I can see that if you draw the chord $e - d$ inside then you must draw $h - i$ outside, and now there is no way to draw the edge from c to g without crossing one or the other chord. So the graph is not planar. The vertices of K_{33} are the six vertices in question, namely e, d, h, i, c, g, and the other two vertices (b and f) count as interrupting vertices.

3l. Use the same idea as in 3g.

7d. A connected planar graph must satisfy $e \leq 3v - 6$. This condition is violated by $e = 14$ and $v = 6$.

7i. Suppose there is a connected graph of this description. Since every region has 4 boundary edges, there are $4r$ dots in a dot-counting argument, and since this is equal to $2e$, we must have $e = 2r$. Since $e - v + 2 = r$, we must have $2r - 12 + 2 = r$ or $r = 10$. So now we know that $e = 20$. We can get this configuration by placing a square within a square within a square and connecting corresponding vertices.

9. To form the line graph of a graph G, place a new vertex in the middle of every edge and connect two of these middle vertices whenever they lie on edges that meet at a vertex. Now erase the original vertices and edges, retaining the middle vertices and the edges that join them. Consider the line graph of K_5. There are 10 edges, and each edge meets 6 other edges. This implies that the line graph has 10 vertices and that each vertex in the line graph has an edge to 6 other vertices in the line graph. So every vertex

degree in the line graph is equal to 6. But I proved in class that every planar graph must have a vertex of degree ≤ 5, so $L(K_5)$ cannot be planar. To answer part (b), consider the graph $G = (V, E)$ where

$$V = \{1, 2, 3, 4, 5, 6\}$$

and

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}.$$

This is a planar graph. Since there are 5 edges, the line graph has 5 vertices. Since every edge meets every other edge, the line graph must be K_5. K_5 is nonplanar.

19. Let G be a connected planar graph. Suppose all vertex degrees are 5 or greater. Then the edge-counting theorem yields $2e \geq 5v$ or $e \geq 2.5v$. On the other hand, $e \leq 3v - 6$. Therefore

$$2.5v \leq e \leq 3v - 6$$

$$2.5v \leq 3v - 6$$
$$6 \leq 0.5v$$
$$12 \leq v$$

hence G must have at least 12 vertices. So if G has fewer than 12 vertices, then it cannot have all vertex degrees 5 or greater, so at least one vertex has degree ≤ 4. In short, all large degrees implies at least 12 vertices, so fewer than 12 vertices implies at least one small degree.

24. See class notes. For part (d), we must count the number of integer solutions to $(d_1 - 2)(d_2 - 2) < 4$. How many non-negative integer solutions to $AB < 4$ are there? $A = 1$ implies $B = 1, 2, 3$. $A = 2$ implies $B = 1$. $A = 3$ implies $B = 1$. So there are 5 solutions.

25. The figure forms a planar graph. The number of vertices is $2l + p$. The p vertices representing line intersections have degree 4 each. The $2l$ vertices on the circle have degree 3 each. So the total vertex degree sum is $6l + 4p$ and the number of edges is $3l + 2p$. So

$$r = e - v + 2 = (3l + 2p) - (2l + p) + 2 = l + p + 2.$$

Therefore the number of interior regions is $l + p + 1$.