Math 375 Exam 1 Spring 2010 Name:

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Notation: Q_n is the n-dimensional hypercube. Vertices are binary strings of length n. An edge connects two vertices when they differ in exactly one bit.

Notation: W_n is the wheel graph on n vertices. Vertices 1 through $n-1$ are arranged in circuit, and vertex n is joined by an edge to each of the other vertices.

1. Let G and H be the graphs depicted in Figure 1. Prove or disprove that they are isomorphic.

Solution: The two graphs have different vertex degree distributions, so they cannot be isomorphic.

2. Prove that a connected planar graph with 5 regions must have at least 8 edges. Hint: use a dot-counting argument.

Solution: Distributing 2 dots per edge, each interior region contributes at least 3 dots, and the exterior region also contributes at least 3 dots (since it wraps around the interior regions). So there are at least 15 dots. Hence $2e \geq 15$, $e \geq 7.5$, which forces $e \geq 8$.

3. Prove that a connected planar graph with 5 regions and 8 edges must have 5 vertices. Draw an example of such a graph.

Solution: $r = e - v + 2$ yields $5 = 8 - v + 2$, which implies $v = 5$. One example is W_5.

4. Prove that Q_3 is planar.

Solution: One of the cycles in Q_3 is $000 - 010 - 110 - 100$. Another one is $001 - 011 - 111 - 101$. Draw the first cycle inside the second cycle, then add the edges $000 - 001, 010 - 011, 110 - 111, 100 - 101$. This is a planar representation of Q_3.

5. Prove that Q_3 is bipartite.

Solution: Color the vertices with an even number of 1s in them red, and color the vertices with an odd number of 1s in them blue. Then all edges connect red edges to blue edges. Hence the graph is bipartite.

6. Assuming that Q_4 is bipartite, prove that it is non-planar by an edge-counting or dot-counting argument.

Solution: If Q_4 is planar, then as a bipartite graph it must satisfy $e \leq 2v-4$. But $e = 32$ and $v = 16$ fails this test.

7. Let G be the planar graph in Figure 7. Prove that it does not have a Hamilton Circuit.

Solution: Suppose it has a Hamilton Circuit. Then

$$2(r_4 - r_4') + 4(r_6 - r_6') + 6(r_8 - r_8') = 0,$$

therefore

$$1(r_4 - r_4') + 2(r_6 - r_6') + 3(r_8 - r_8') = 0.$$

In modulo 2 arithmetic, this says

$$1(r_4 + r_4') + 1(r_8 + r_8') \equiv 0.$$

However, there are an even number of regions with 4 edges and an odd number of regions with 8 edges, therefore $r_4 + r_4' \equiv 0$ and $r_8 + r_8' \equiv 1$. This implies

$$1(r_4 + r_4') + 1(r_8 + r_8') \equiv 1.$$

This contradiction implies that there is no Hamilton Circuit.

8. Compute $\chi_v(W_6)$ and $\chi_e(W_6)$.

Solution: The presence of K_3 in W_6 implies that $\chi_v(W_6) \geq 3$. But 3 colors are not possible, while 4 are, so $\chi_v(W_6) = 4$. The presence of a degree 5 vertex in W_6 implies that $\chi_e(W_6) \geq 5$. Since 5 colors are possible, $\chi_e(W_6) = 5$.