Week 8 Lectures: Sections 4.6 and 4.7

Linearly dependent vectors: The vectors v_1 through v_k are linearly dependent if and only there is a non-trivial linear combination (at least one non-zero coefficient) which produces the 0 vector. Example: $(1,2)$, $(1,3)$, $(1,4)$, and $(1,5)$ are linearly dependent in $\mathbf{R}^2$ because

$$3 \cdot (1,2) - 5 \cdot (1,3) + 1 \cdot (1,4) + 1 \cdot (1,5) = 0.$$

Theorem: *If $m > n$, then any m vectors in $\mathbf{R}^n$ are linearly dependent.*

Proof: Start with our example above. These are 4 vectors in $\mathbf{R}^2$, and they are linearly dependent. We can actually prove that they are linearly dependent without producing the coefficients: Solving

$$a \cdot (1,2) + b \cdot (1,3) + c \cdot (1,4) + d \cdot (1,5) = 0,$$

we see that there must be exactly 2 equations (because there are 2 coordinates) and there are exactly 4 variables (a, b, c and d). So we get a system of 2 equations in 4 variables in which the constants on the right-hand side are both 0. In principle, if we set up the augmented matrix and row-reduce, there will have to be at least two non-leading variables. These can take on the value 1, so these are the non-zero coefficient we are looking for.

In detail:

$$a + b + c + d = 0$$

$$2a + 3b + 4c + 5d = 0$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right].$$

In general, the argument is this: Given v_1 through v_m in $\mathbf{R}^n$, the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

leads to n equations (because there are n coordinates in each vector) for the m variables α_1 through α_m. Since $m > n$, the augmented matrix when row-reduced will have at most n leading terms, leaving at least $m - n > 0$ non-leading terms. We can set non-leading terms equal to 1, and these are the non-zero coefficients we are looking for.

The meaning of linear dependence: Let v_1 through v_k be linearly dependent. Let $V = \text{span}\{v_1, \ldots, v_k\}$. Then there we can discard one of the vectors v_i without changing the span.

Example: Above, we showed that $(1,2)$, $(1,3)$, $(1,4)$ and $(1,5)$ are linearly dependent. Moreover, we showed that

$$3 \cdot (1,2) - 5 \cdot (1,3) + 1 \cdot (1,4) + 1 \cdot (1,5) = 0,$$

which implies that

$$(1,3) = \frac{3}{5}(1,2) + \frac{1}{5}(1,4) + \frac{1}{5}(1,5).$$

Therefore

$$(1,3) \in \text{span}\{(1,2),(1,4),(1,5)\}.$$

This implies

$$\text{span}\{(1,2),(1,3),(1,4),(1,5)\} = \text{span}\{(1,2),(1,4),(1,5)\}.$$

Reason: it is clear that

$$\text{span}\{(1,2),(1,4),(1,5)\} \subseteq \text{span}\{(1,2),(1,3),(1,4),(1,5)\}.$$

We also know that

$$(1,2),(1,3),(1,4),(1,5) \in \text{span}\{(1,2),(1,4),(1,5)\}.$$

Therefore

$$\text{span}\{(1,2),(1,3),(1,4),(1,5)\} \subseteq \text{span}\{(1,2),((1,4),(1,5)\}.$$

It should be clear that we can continue to eliminate vectors until we boil it down to some minimal subset. To continue with our example, we discarded $(1,3)$, leaving $(1,2)$, $(1,4)$, $(1,5)$. These three must still be linearly dependent, because these are 3 vectors in $\mathbf{R}^2$. In fact, we have

$$1 \cdot (1,2) - 3 \cdot (1,4) + 2 \cdot (1,5) = (0,0),$$

$$(1,2) = 3 \cdot (1,4) - 2 \cdot (1,5),$$

therefore

$$(1,2) \in \text{span}\{(1,4),(1,5)\}.$$

Hence

$$\text{span}\{(1,2),(1,3),(1,4),(1,5)\} = \text{span}\{(1,2),(1,4),(1,5)\} = \text{span}\{(1,4),(1,5)\}.$$

Note that the vectors $(1,4)$ and $(1,5)$ are NOT linearly dependent. This leads us to the following definition:

Linearly independent vectors: A set of vectors v_1 through v_k such that the only solution to

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

To continue with our running example, we started with

$$V = \text{span}\{(1,2),(1,3),(1,4),(1,5)\}$$

and boiled this down to

$$V = \text{span}\{(1,4),(1,5)\}.$$

The set of vectors $\{(1,4),(1,5)\}$ is linearly independent. This gives rise to the following definition:

Basis for a subspace: A set of vectors which span the subspace and which are linearly independent.

A basis for the subspace V in our running example is the set $\{(1,4),(1,5)\}$. A subspace can have many different bases. Another basis for our V is the set $\{(1,2),(1,3)\}$. It turns out that every basis for a subspace has the same size.

Theorem: *Let V be a subspace. Then every finite basis for V has the same size.*

Proof: Suppose that v_1 through v_a is one basis for V, and that w_1 through w_b is another basis for V, where $a < b$. We will obtain a contradiction as follows:

Since v_1 through v_a is a basis for V, we know that

$$V = \text{span}\{v_1, \ldots, v_a\}.$$

Therefore every vector in V is a linear combination of v_1 through v_a. In particular, each w_i is. We can write

$$w_1 = \alpha_{11}v_1 + \alpha_{12}v_2 + \cdots + \alpha_{1a}v_a,$$

$$w_2 = \alpha_{21}v_1 + \alpha_{22}v_2 + \cdots + \alpha_{2a}v_a,$$

$$\cdots$$

$$w_b = \alpha_{b1}v_1 + \alpha_{b2}v_2 + \cdots + \alpha_{ba}v_a.$$

We will now create vectors in $\mathbf{R}^a$ using these coefficients:

$$x_1 = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1a}),$$

$$x_2 = (\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2a}),$$

$$\cdots$$

$$x_b = (\alpha_{b1}, \alpha_{b2}, \cdots, \alpha_{ba}).$$

Since these are $b > a$ vectors in $\mathbf{R}^a$, we know they must be linearly dependent. Therefore there is a solution to

$$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_b x_b = 0$$

in which at least one of the coefficients $\beta_i \neq 0$. However, we can see that

$$\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_b w_b = 0.$$

This is impossible, because w_1 through w_b are linearly independent. Therefore we cannot have two different sized bases.

Dimension of a subspace: The size of any basis for the subspace.

Our space $V = \text{span}\{(1,2),(1,3),(1,4),(1,5)\}$ has dimension 2.

Dimension of $\mathbf{R}^n$: Must be n, because a basis for $\mathbf{R}^n$ is $(1,0,\ldots,0)$, $(0,1,0,\ldots,0)$, ..., $(0,0,\ldots,1)$.

Dimension of any subspace V of $\mathbf{R}^n$: must be $\leq n$. Reason: Suppose v_1 through v_k is a basis for V. Then $\dim(V) = k$. We know that if $k > n$ then the vectors v_1 through v_k are linearly dependent. Therefore $k \leq n$.

Theorem: *If v_1 through v_k are linearly independent and $v \notin \text{span}\{v_1,\ldots,v_k\}$, then v_1 through v_{k+1} are linearly independent, where $v_{k+1} = v$.*

Proof: Suppose

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k+1} v_{k+1} = 0.$$

We wish to prove that all the coefficients are equal to zero. We know that α_{k+1} must be equal to zero, otherwise we could solve for v_{k+1} in terms of v_1 through v_k, which contradicts our hypothesis that $v \notin \text{span}\{v_1,\ldots,v_k\}$. Therefore

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0,$$

which implies that all the other coefficients are zero since v_1 through v_k are linearly independent.

Theorem: *Let v_1 through v_n be linearly independent vectors in $\mathbf{R}^n$. Then they form a basis for $\mathbf{R}^n$. In particular, they span $\mathbf{R}^n$.*

Proof: Suppose

$$\text{span}\{v_1,\ldots,v_n\} \neq \mathbf{R}^n.$$

Then we can find $v \notin \text{span}\{v_1,\ldots,v_n\}$, which implies that v_1 through v_{n+1} are linearly independent in $\mathbf{R}^n$, where $v_{n+1} = v$. Contradiction. Therefore

$$\text{span}\{v_1,\ldots,v_n\} = \mathbf{R}^n.$$

Theorem: *Let v_n through v_n be vectors which span $\mathbf{R}^n$. Then they must be linearly independent.*

Proof: We have

$$\mathbf{R}^n = \text{span}\{v_1,\ldots,v_n\}.$$

If v_1 through v_n are not linearly independent, then we know that we can continue to throw some of them away without changing the span until we boil down to a minimal set which is linearly independent. We can then say

$$\mathbf{R}^n = \text{span}\{v_1,\ldots,v_k\}$$

where v_1 through v_k are linearly independent and $k < n$. This says that $\mathbf{R}^n$ has dimension $k < n$. Contradiction. Therefore v_1 through v_n must be linearly independent.

Method for constructing a basis for a subspace V of $\mathbf{R}^n$: Choose $v_1 \neq 0$ in V. If $V = \text{span}\{v_1\}$, then v_1 forms a basis for V since one non-zero vector creates a linearly independent set. Otherwise, we can choose $v_2 \notin \text{span}\{v_1\}$. Then we know that v_1 and v_2 are linearly independent. If $V = \text{span}\{v_1, v_2\}$, then v_1 and v_2 form a basis for V. If $V \neq \text{span}\{v_1, v_2\}$, then we can choose $v_3 \notin \text{span}\{v_1, v_2\}$. Keep on going, enlarging our set of linearly independent vectors in V. This process has to stop, because the maximum number of linearly independent vectors we can find in $\mathbf{R}^n$ is n. So when we stop we have

$$V = \text{span}\{v_1, \ldots, v_k\},$$

and v_1 through v_k form a basis for V. Note that this proof implies $\dim(V) \leq n$.

Identification of a subspace of dimension k with $\mathbf{R}^k$: Let V be a subspace with dimension k. Then it has a basis v_1 through v_k, and these vectors span V. If $w \in V$, then there are unique coefficients α_1 through α_k such that

$$w = \alpha_1 v_1 + \cdots + \alpha_k v_k.$$

Reason: suppose

$$w = \beta_1 v_1 + \cdots + \beta_k v_k$$

is another representation of w. Equating, we obtain

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \beta_1 v_1 + \cdots + \beta_k v_k,$$

$$(\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_k - \beta_k) v_k = 0,$$

which implies $\alpha_1 - \beta_1 = \cdots = \alpha_k - \beta_k = 0$ by linear independence. Therefore $\alpha_i = \beta_i$ for all i.

Theorem: *If $V \subseteq W$ are subspaces and W has dimension n, then dim$(V) \leq n$.*

Proof: We can identify W with $\mathbf{R}^n$. Under this identification, V can be regarded as a subspace of $\mathbf{R}^n$. Therefore it must have dimension $\leq n$.

Theorem: *Let $V \subseteq W$ be subspaces, and assume dim$(W) = n$. If dim$(V) = n$ then $V = W$.*

Proof: Suppose $V \neq W$. Then there is a vector w in W which does not belong to V. If v_1 through v_n form a basis for V, then v_1 through v_{n+1} are linearly independent in W, where $v_{n+1} = w$. Set $V' = \text{span}\{v_1, \ldots, v_{n+1}\}$. Then $V' \subseteq W$ and $\dim(V') = n + 1 > n$. This contradicts the previous theorem. Therefore $V = W$.

Using the ideas above, we can also prove that if V is a subspace of dimension n and v_1 through v_n are vectors in V, then they span V if and only if they are linearly independent. We proved this for $V = \mathbf{R}^n$. If all we know is $\dim(V) = n$, then we can make the identification of V with $\mathbf{R}^n$ and proceed from there.

Exercises:

Section 4.6, 3bd, 4bf, 6bc, 10b, 11bd, 14bc

Section 4.7: 3cd, 6ceg, 7, 14, 15bd, 17bce, 20bd