Section 3.1: Nonseparable Graphs

Cut vertex of a connected graph $G$: A vertex $x \in G$ such that $G - x$ is not connected.

Theorem 3.1, p. 57: Every connected graph $G$ with at least 2 vertices contains at least 2 non-cut vertices.

Proof: By strong induction $v$. Clear for $v = 2$. For $v > 2$, true if there are no cut vertices. If $x$ is a cut vertex, write $G = x + G_1 + G_2 + \cdots + G_k$. Each single-vertex $G_i$ is a non-cut vertex. Each non-trivial $G_i$ has at least two non-cut vertices, one of which is non-adjacent to $x$; the other one is a non-cut vertex of $G$. Hence there are at least $k$ non-cut vertices. \qed

Nonseparable graph: Connected, at least 2 vertices, and no cut vertices.

Theorem: Let $G$ be a connected graph with 3 or more vertices. The following statements are equivalent:

1. For each $(x, y)$ in $V \times V$ there is a cycle $C_{xy}$ that contains both $x$ and $y$.
2. $G$ is in separable.
3. For each $(e, f) \in E \times E$ there is a cycle $C_{ef}$ that contains both $e$ and $f$.

Proof: (1) implies (2): Let $x \in V$ be given. For each $y, z$ in $G - x$ there is a $C_{yz}$ in $G$ that contains both $y$ and $z$. Since $C_{yz}$ yields two internally disjoint $yz$ paths, the removal of $x$ disconnects at most one of these, leaving a $yz$ path in $G - x$. Hence $G - x$ is connected. Since $G$ has at least 2 vertices, $G$ is nonseparable.

(2) implies (3): Suppose $G$ is inseparable. Let $(e, f) \in E \times E$ be given. We will show that there is a cycle through both $e$ and $f$ by strong induction on the shortest distance $d$ between a vertex in $e$ and a vertex in $f$.

If $d = 0$ and $e$ and $f$ have the same endpoints $x$ and $y$, let $z$ be a third vertex, let $P_x$ be a $xz$ path in $G - y$, and let $P_y$ be an $yz$-path in $G - x$. Let $y_0$ be the vertex along $P_y$ closest to $P_y$ that belongs to $P_x$. An $xy$ path $Q$ that excludes the edge $xy$ is to follow $P_x$ to $y_0$, then follow $P_y$ back to $y$. The desired cycle is $Q + xy$.

If $d = 0$ and $e$ and $f$ share a single endpoint $x$, let $y$ and $z$ be the other two endpoints. Then $P_{yz} + zx + xy$ is a cycle through $e$ and $f$ where $P_{yz}$ is a $yz$-path in $G - x$. 

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If \( d > 0 \), write \( e = xy \) and \( f = XY \) and assume without loss of generality that \( d(x, X) \) is the shortest distance between \( e \) and \( f \). Let \([x, X]\) be a geodesic with last edge \( ZX \). Then \( d(xy, ZX) < d(xy, XY) \), and there is a cycle \( C \) through \( xy \) and \( ZX \). Let \( R \) be the shortest path in \( G - X \) from \( Y \) to a vertex in \( C \). Then \( R \) is internally disjoint from \( C \), otherwise there is a shorter path to \( C \). Follow \( R \) from \( Y \) to \( C \), then \( C \) to \( X \) in the direction that passes through \( xy \) first, then take the edge from \( X \) to \( Y \). This is a cycle incorporating \( e \) and \( f \). See the figure below.

(3) implies (1): Let \( x \) and \( y \) be vertices in \( G \). Since \( G \) is connected, there are edges \( e \) and \( f \) containing \( x \) and \( y \). The cycle \( C_{ef} \) contains both \( x \) and \( y \). The cycle \( C_{ef} \) contains both \( x \) and \( y \). □

**Block:** Maximal nonseparable subgraph \( B \) of a connected graph \( G \), in the sense that \( B \) is not the proper subgraph of any nonseparable subgraph of \( G \).

**Remark:** The next theorem proves among other things that the blocks of a connected graph partition the edges of the graph. After the next few results we will describe an efficient method for finding the blocks in a graph.

**Theorem (elementary properties of blocks):**

(1) Distinct blocks have at most one vertex in common.

(2) Every nonseparable subgraph is a subgraph of unique block.

(3) A vertex is a cut vertex if and only if it belongs to two or more blocks.

**Proof:**

(1) If \( x \in B \cap C \) and \( y \in B \cap C \) and \( x \neq y \) then removing any \( v \in B \cup C \) leaves \( B - v \) and \( C - v \) connected and leaves either \( x \) or \( y \) intact in \((B \cup C) - v\). Hence \( B \cup C \) is inseparable, which forces \( B = C \) by maximality. So distinct blocks cannot share two vertices.
(2) First observe that when $H$ is a non-separable subgraph of $G$, so is $G[H]$, the subgraph induced by $V_H$. Now let $W$ be a subset of $V$ of maximum size such that $V_H \subseteq W$ and $G[W]$ is nonseparable. We claim that $G[W]$ is a block. If not, it is a proper subgraph of a nonseparable subgraph $K$, hence of the nonseparable subgraph $G[K]$, which is impossible since $|W| < |V_K|$.

To prove uniqueness, suppose $H$ is a subgraph of blocks $B$ and $C$. Since $H$ has an edge $xy$, $B$ and $C$ both contain $x$ and $y$. Since blocks cannot share more than one vertex, $B = C$.

(3) Let $x \in B \cap C$ where $B \neq C$. Since $B = G[V_B]$ and $C = G[V_C]$, $V_B \nsubseteq V_C$ and $V_C \nsubseteq V_B$ by maximality of blocks. Hence we can choose $b \in B - C$ and $c \in C - B$, and we must have $b \neq x$ and $c \neq x$. If $x$ is not a cut vertex, then $G - x$ is connected, hence there is a $b, c$ path $P$ that excludes $x$. However, this makes $B \cup P \cup C$ nonseparable (see diagram below), a contradiction. Therefore $x$ is a cut vertex.

 Conversely, if $x$ is a cut vertex, write $G - x = G_1 + G_2 + \cdots + G_k$. Then for each $i$ there is an edge $xx_i$ where $x_i \in G_i$. Let $B_i$ be the block containing $xx_i$. Then $B_i - x$ is connected and must be a subgraph of $G_i$ since it contains $x_i$. Hence the $B_i$ are distinct and $x$ lives in $k$ distinct blocks. 

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\textbf{Theorem (recognizing a block decomposition, version I):} Let $G_1, \ldots, G_r$ be a collection of non-separable subgraphs of a connected graph $G$ that partitions the edge set of $G$. If every cycle of $G$ is a subgraph of some $G_i$, then $G_1, \ldots, G_r$ is the block-decomposition of $G$.

\textbf{Proof:} Let $B$ be a block of $G$. If $B$ consists of a single edge then $B \subseteq G_i$ for some $i$, hence by maximality $B = G_i$. If $B$ contains more than one edge, fix one edge $e$, and let $i$ be the unique index such that $e \in G_i$. Then for every other edge $f$ of $B$ there is a cycle $C_{ef}$ that contains both $e$ and $f$. Since $C_{ef}$ is a subgraph of some $G_j$ and $e \in G_j$, $j = i$. Therefore every edge of $B$ is a subgraph of $G_i$, which implies that $B$ is a subgraph of $G_i$, which implies $B = G_i$. \hfill \square
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**Fundamental Cycles:** Let $G$ be connected and let $T$ be a spanning tree of $G$. For each edge $xy$ in $G$ but not in $T$, there is a unique $xy$ path $P_{xy}$ in $T$. $P_{xy} + xy$ is called a fundamental cycle with respect to $T$.

**Theorem (recognizing a block decomposition, version II):** Let $G_1, \ldots, G_r$ be edge-disjoint non-separable subgraphs of a connected graph $G$. If $T$ is a spanning tree of $G$ and every fundamental cycle of $T$ is a subgraph of some $G_i$ then $G_1, \ldots, G_r$ is the block-decomposition of $G$.

**Proof:** We will prove that every cycle $C$ is a subgraph of $G_i$ for some $i$ by induction on $|C - T|$. By the previous theorem, this implies that $G_1, \ldots, G_r$ is the block-decomposition of $G$.

1. $|C - T| = 1$: $C$ is a fundamental cycle with respect to $T$ and is a subgraph of some $G_i$.
2. $|C - T| > 1$: Fix $ab \in C - T$. Write $C = abx_1 \cdots x_na$.

Let $P = ay_1 \cdots y_m b$ be a path in $T$. There are two cases to consider.

- **Case 1:** $\{y_1, \ldots, y_m\} \cap \{x_1, \ldots, x_n\} = \emptyset$. Then two cycles in $G$ are $D = ay_1 \cdots y_m ba$ and $E = ay_1 \cdots y_m bx_1 \cdots x_n$.

Since $D$ is a fundamental cycle, $D$ is a subgraph of some $G_i$. Since $|E - T| = |C - T| - 1$,

$E$ is a subgraph of some $G_j$ by the induction hypothesis. Since $D$ and $E$ share the edge $ay_1$, $G_i = G_j$. Therefore $C \subseteq G_i$. 

![Diagram](image-url)
Case 2: \( \{y_1, \ldots, y_m\} \cap \{x_1, \ldots, x_n\} \neq \emptyset \). Let \( i \) be least such that \( y_i = x_j \) for some \( j \). Then
\[
D = ay_1 \cdots y_ix_{j+1} \cdots x_na
\]
and
\[
E = ay_1 \cdots y_ix_{j-1} \cdots x_1ba.
\]
Since \( T \) is acyclic, there has to be a non-tree edge in \( D \) that belongs to \( C \).
Since \( ab \not\in D \),
\[
1 \leq |D - T| < |C - T|.
\]
Since \( ab \in E \),
\[
1 \leq |E - T| < |C - T|.
\]
As in Case 1, this implies \( C \) is a subgraph of some \( G_i \).

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**Remark:** The complete graph \( K_n \) has roughly \( 2^n \) cycle subgraphs but only roughly \( \frac{n^2}{2} \) fundamental cycles, so the second version of this theorem is much more efficient than the first.

**Theorem (gluing nonseparable subgraphs):** If \( G \) is a connected graph, \( C \) is a cycle in \( G \), and \( H_e \) is a non-separable subgraph containing \( e \) for each edge \( e \) in \( C \), then the graph union \( H = \bigcup_{e \in C} H_e \) is non-separable.

**Proof:** Let \( x \) be an arbitrary vertex in \( H \). We must show that \( H - x \) is connected. Let \( a \) and \( b \) be vertices in \( H - x \). Then \( a \) is a vertex in \( H_e - x \) and \( b \) is a vertex in \( H_f - x \) for some \( e \) and \( f \). Let \( e_x \) be one of the endpoints of \( e \) not equal to \( x \), and let \( f_x \) be one of the endpoints of \( f \) not equal to \( x \). By non-separability of \( H_e \) and \( H_f \), there is an \( ae_x \) path in \( H_e - x \) and a \( bf_x \) path in \( H_f - x \). Since \( e_x \) and \( f_x \) belong to \( C - x \) and \( C - x \) is connected,
Algorithm for finding the blocks in a graph: Let $T$ be a spanning tree. Let $L_0$ be the list of $K_2$ subgraphs (edges) of $G$. Then $L_0$ partitions the edges of $G$ into non-separable subgraphs. Having defined the edge partition $L_{i-1}$, consisting of nonseparable subgraphs containing fundamental cycles $T_{e_i}$ through $T_{e_{i-1}}$, let $T_{e_i}$ be the $i^{th}$ fundamental cycle. Form the union $H_i$ of all the subgraphs in $L_{i-1}$ that have an edge in common with $T_{e_i}$, and let $L_i$ be the set consisting of $H_i$ and the remaining elements of $L_{i-1}$. Then $L_i$ partitions the edges of $G$ into non-separable subgraphs, and the fundamental cycles $T_{e_1}$ through $T_{e_i}$ each belong to some subgraph in $L_i$. Keep on going, running through all $r$ fundamental cycles. Then $L_r$ is the block decomposition of $G$.

Example: Apply this algorithm to the graph on 60 (Figure 3.2).

Block-cut graph: Let $G$ be a connected graph with cut vertices $x_1, \ldots, x_a$ and blocks $B_1, \ldots, B_b$. Form the bipartite graph with vertices

$$x_1, \ldots, x_a, B_1, \ldots, B_b$$

and an edge of the form $x_iB_j$ if and only if $x_i \in B_j$.

Lemma: Let $G$ be a connected graph with at least two vertices, let $k \geq 1$ be given, and let $x_0, x_1, \ldots, x_{k+1}$ be a path in $G$ in which none of the vertices $x_1$ through $x_k$ is a cut vertex. Then there is a block of $G$ containing $x_0$ through $x_{k+1}$.

Proof: For each $i \in \{1, \ldots, k+1\}$ let $B_i$ be a block containing the edge $x_{i-1}x_i$. Then for $1 \leq i \leq k$, $x_i \in B_i \cap B_{i+1}$, hence $B_i = B_{i+1}$ for $1 \leq i \leq k$. This implies $B_1 = \cdots = B_{k+1}$. Call this common block $B$. Then $B$ contains $x_0$ through $x_{k+1}$.

Lemma: Let $G$ be a connected graph with at least two vertices. Then for every pair of blocks $A$ and $B$, and every path $P$ that begins in a non-cut vertex $a \in A$ and ends in a non-cut vertex $b \in B$, there is an $AB$ walk $W_P$ in the block-cut graph that contains every cut vertex appearing among the internal vertices of $P$.

Proof: By strong induction on the number of cut vertices internal to $P$. If there are none, then $P$ belongs in a single block $C$. Since $a$ and $b$ belong to unique blocks, $A = C = B$. So we can set $W_P = A$. Assume that $W_P$ can be
constructed when $P$ contains up to $k$ internal cut vertices. Now consider $P$ with $k + 1$ internal cut vertices. Let $x$ be any of these, and say that $x$ belongs to block $C$. Then $P$ decomposes into walks $Q = P[a, x]$ and $R = P[x, b]$, and each has $\leq k$ internal cut vertices. By the induction hypothesis there is an $AC$ walk $W_Q$ and a $CB$ walk $W_R$ in the block-cut graph that contains the internal cut vertices of $Q$ and $R$. The walk $W_Q, x, W_R$ is the desired $AB$ walk in the block-cut graph.

**Corollary:** The block-cut graph is connected.

**Proof:** Given blocks $A$ and $B$, choose a non-cut vertex $a \in A$ and a non-cut vertex $b \in B$, and choose an $ab$ path $P$. Then $W_P$ is an $AB$ walk in the block-cut graph. Given a cut vertex $x$ and a block $B$, pick a block $A$ that $x$ belongs to, then follow the edge from $x$ to $A$ and the walk from $A$ to $B$ in the block-cut graph to obtain a walk from $x$ to $B$ in the block-cut graph. Given a cut vertex $x$ and cut vertex $y$, walk from $x$ to $A$, then from $A$ to $B$, then from $B$ to $y$ in the block-cut graph, where $x$ is in block $A$ and $y$ is in block $B$.

**Theorem:** The block-cut graph is a tree.

**Proof:** We’ve proved connectivity above. Next, suppose there is a cycle in the block-cut graph of the form

$$x_0, B_0, x_1, B_1, \ldots, x_n, B_n, x_0.$$ 

Since every pair of consecutive cut-vertices along this cycle live in the same block, there is a path joining them in that block. Concatenating these paths yields a closed walk in $G$, which yields a cycle $C$ in $G$. Glueing together the blocks contributing edges to $C$ yields a nonseparable subgraph which (1) lives in a single block $B$ and (2) has edges from at least two different paths. This is impossible, because $B$ contains edges from two different paths, hence two different blocks, a contradiction. So there cannot be a cycle in the block-cut graph.

**Remark:** It is possible for two vertices $a$ and $b$ in a connected graph to be joined by a path containing no internal cut vertices and another path with an internal cut vertex. Just consider the graph with a cycle through $a, b, x$ and a fourth edge $xc$. But when $a$ and $b$ are in different blocks, this cannot happen, as the corollary below shows.
Corollary: Let $G$ be a connected graph with at least two vertices, and let $a$ and $b$ be vertices in different blocks of $G$. Then the internal vertices of every path $ab$ walk $P$ of $G$ contains the same subset of cut vertices of $G$.

Proof: Given any $ab$ path $x_0, x_1, \ldots, x_r$ in $G$, let $B_i$ be a block containing the edge $x_{i-1}x_i$ for $1 \leq i \leq r$. Then there is a walk in the block-cut graph from $B_1$ through $B_r$ that contains all the cut vertices among $x_1, \ldots, x_{r-1}$. If there is a second path $ab$ path $y_0, y_1, \ldots, y_s$, it yields a second block-cut graph $C_1, \ldots, C_s$, defined similarly. If $B_1 = C_1$ and $B_r = C_s$, we obtain two $B_1B_r$ walks in the block-cut graph with different internal vertex sets, which implies a cycle in the block-cut graph, which is impossible. If $B_1 \neq C_1$ and $B_r = C_s$, we obtain two $aB_r$ walks in the block-cut graph with internal vertex sets, which yields another contradiction. The other two cases also yield a contradiction. Therefore every $ab$ path in $G$ contains the same internal vertices that are cut vertices.

Lemma: Let $G$ be a connected graph and let $a$ and $b$ be distinct vertices in $G$. If $a$ and $b$ do not live in the same block then $G$ has a cut vertex $x$ such that $a$ and $b$ are in different connected components of $G - x$.

Proof: Let $P$ be an $ab$ path. Then $P$ contains at least one internal vertex $x$ that is a cut vertex, otherwise $a$ and $b$ live in the same block. Hence every $ab$ path contains $x$. Therefore $a$ and $b$ live in different components of $G - x$. □

Theorem 3.9, p. 61: When $G$ is a connected graph with at least two vertices, every central vertex lives in the same block of $G$.

Proof: Let $a$ and $b$ any pair of vertices that do not live in the same block. We will show that at least one of them is noncentral. Let $x$ be a cut vertex such that $a$ and $b$ are in different connected components of $G - x$. Let $z$ be a vertex in $G$ farthest from $x$. $z$ cannot be simultaneously in the component of $G - x$ containing $a$ and the component of $G - x$ containing $b$. If $z$ is not in the same component as $a$, then since $d(z, a) = d(z, x) + d(z, x)$, $ecc(a) \geq d(z, a) > d(z, x) = ecc(x)$, which implies that $a$ is not central. Similarly if $z$ is not in the same component as $b$ then $b$ is not central. □

End block: A block containing exactly one cut vertex, i.e. having degree 1 in the block-cut graph.

Theorem 3.7, p. 60: Every connected graph containing cut vertices has at least two end blocks.
Proof: The block-cut graph has at least two leaves, i.e. degree 1 vertices. These must be blocks, because cut vertices have degree at least two. □

Theorem 3.8, p. 61: Let $G$ be a connected graph with cut vertices. Then $G$ contains a cut-vertex such that at most one of the blocks it belongs to is not an end block.

Proof: Let $B$ be a leaf in the block-cut graph. Let $x$ be the unique cut-vertex it is adjacent to in the block-cut graph. This is the desired cut-vertex. □