Week 6 Lectures

Sections 7.5, 7.6

Section 7.5: Surface Area of Revolution

Surface Area of Cone: Let C be a circle of radius r. Let P_n be an n-sided regular polygon of perimeter p_n with vertices on C. Form a cone C_n of slant length l by glueing together n iscoseles triangles with sides of length $l, l, \frac{p_n}{n}$. Given that each triangle has area $\frac{1}{2}\frac{p_n}{n}\sqrt{l^2 - \frac{p_n^2}{4n^2}}$, the total surface area of the cone is $\frac{1}{2}p_n\sqrt{l^2 - \frac{p_n^2}{4n^2}}$. The cone C_n is an approximation of the right circular cone corresponding to a circle of radius r and slant length l. Given that

$$\lim_{n\to\infty} \frac{1}{2}p_n\sqrt{l^2 - \frac{p_n^2}{4n^2}} = \pi r l,$$

the surface area of the cone with circular base is $\pi r l$.

Surface Area of Conic Frustrum: To obtain the surface area of a conic frustrum with slant length l and radii r and R, imagine subtracting two cones, the small one with slant length l_1 and radius r_1, and the large one with slant length l_2 and radius r_2. Then the net surface area is $\pi r_2 l_2 - \pi r_1 l_1$. Given the relationships

$$\frac{l_1}{l_2} = \frac{r_1}{r_2}$$

and

$$l_2 - l_1 = l,$$

the net surface area (after simplification) is $\pi l(r_1 + r_2)$.

Surface Area of Revolution: Let $y = f(x)$ be a curve from $x = a$ to $x = b$. Rotating about the x-axis yields a surface of revolution. Approximating the curve by line segments with slope $f'(x_i)$, we obtain conic frustrums with slant length $\sqrt{1 + f'(x_i^*)^2}\Delta x$ and radii $f(x_{i-1})$ and $f(x_i)$. Summing, we obtain the approximation

$$\sum_{i=1}^{N} \pi\sqrt{1 + f'(x_i^*)^2}(f(x_{i-1}) + f(x_i))\Delta x.$$

Using the further approximation $f(x_{i-1}) + f(x_i)) \approx 2f(x_i^*)$, we obtain the Riemann sum

$$\sum_{i=1}^{N} 2\pi f(x_i^*)\sqrt{1 + f'(x_i^*)^2}\Delta x.$$

This yields

$$\text{surface area } = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\ dx.$$

For example, using $f(x) = \frac{x^2}{2}$, $x \in [0, 1]$, we obtain

$$\int_0^1 x^2\sqrt{1 + x^2}\ dx = \int_0^{\frac{\pi}{4}} \tan^2\theta \sec^3\theta\ d\theta = \int_0^{\frac{\pi}{4}} \sec^5\theta - \sec^3\theta\ d\theta =$$

$$\int_0^{\frac{\pi}{4}} (\sec^6\theta - \sec^4\theta)\cos\theta\ d\theta = \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{(1-u^2)^3} - \frac{1}{(1-u^2)^2}\ du =$$

$$I_3(u) - I_2(u)|_0^{\frac{\pi}{4}} = \frac{1}{16}\left(6\sqrt{2} - \log\left(2\sqrt{2} + 3\right)\right).$$

Homework: Section 7.5, problems 7, 9, 11, 13, 25.

Section 7.6: Applications to Physics and Engineering

Things that can be measured:

distance (d), time (t), mass (m).

Things that can be calculated:

area (A), velocity (v), acceleration (a), force (F), pressure (P), work (W).

English Units:

d ft, t sec, v ft/sec, a ft/sec^2, F lbs, $P = F/A$ lbs/ft^2, $W = Fd$ ft-lbs.

Metric Units:

d m, t sec, m kg, v m/sec, a m/sec^2, $F = ma$ N, $P = F/A$ N/m^2,
$W = Fd$ J.

Note that N is short for netwons (kg m/sec^2) and J is short for Joules (newton-meters or kg m^2/sec^2).

Constants:

Gravity near earth causes constant acceleration of 32 ft/sec^2 and 9.8 m/sec^2.

Water has weight density 62.5 lbs/in^3 and mass density 1000 kg/m^3.

Work calculations:

If you lift an object of weight 23 ounces (force) through 75 inches (distance), then the work done is

$$W = Fd = \frac{23}{16} \cdot \frac{75}{12} = 8.98 \text{ ft lbs.}$$

If you lift an object of mass 700 g through 250 cm, the work done is

$$W = Fd = mad = \frac{700}{1000} \cdot 9.8 \cdot \frac{250}{100} = 17.15 \text{ J.}$$

A Variable Work Calculation: Example 3, page 400. A 200 pound rope that is 100 feet long is suspended from the top of a building. Find the work done in pulling up the rope to the top of the building, discounting other forces.

Solution: Think of the rope as being partitioned into N pieces of length Δx feet. We will calculate the work done to lift each segment, then add.

Each segment weighs $2\Delta x$ pounds, and the i^{th} segment from the top is lifted through x_i^* feet, so the $2x_i^*\Delta x$ foot pounds of work is done, for a total of

$$\sum_{i=1}^{N} 2x_i^*\Delta x$$

foot pounds. Since x_i^* is varying in $[0, 100]$, the exact amount the work is

$$W = \int_0^{100} 2x \; dx$$

foot-pounds.

The Leaky Bucket Problem: Imagine that the rope above is supporting a bucket of 50 gallons of water, that an empty bucket weighs 10 pounds, that we pull up the bucket at a rate of 10 feet per second, and that at the moment we start pulling the bucket leaks water at a rate of 2 gallons per second. Calculate the work done in pulling up the rope and the bucket and the water in the bucket.

Solution: We will just calculate the amount of work to pull up the bucket and the water in it, then add to the previous answer. It will take 10 seconds to pull up the bucket. Think of time as being partitioned into N subintervals of Δt seconds. We will calculate the work done in each subinterval of time.

From time t_i to time t_{i+1} we have lifted the bucket $10\Delta t$ feet. Choosing any t_i^* in this interval, the bucket and water weighs approximately $10 + 50 - 2t_i^* = 60 - 2t_i^*$ pounds, so in this time interval we have done approximately $(600 - 20t_i^*)\Delta t$ foot-pounds of work. Total approximate work done is

$$\sum_{i=1}^{N} (600 - 20t_i^*)\Delta t.$$

Since t_i^* is varying in $[0, 10]$, the exact amount of work done is

$$W = \int_0^{10} 600 - 20t \; dt.$$

Pumping Water out of a Tank Formed by a Volume of Revolution: Imagine that a tank of water is formed by revolving the region bounded by $y = x^4$ and $y = 3$ about the y-axis. The tank is filled with water, and the water is to be pumped out. How much work does this take, assuming that the units along the x and y axes are given in feet?

Solution: We will partition the tank into slices of width Δy and calculate the work done to pump each slice out. The slice that extends from y_{i-1} to y_i can be approximated by a washer that has a thickness of Δy feet, a cross-sectional radius of approximately x_i^* feet corresponding to a value of $y_i^* \in [y_{i-1}, y_i]$, a cross-sectional area of approximately $\pi(x_i^*)^2 = \pi\sqrt{y_i^*}$ square feet, a volume of approximately $\pi\sqrt{y_i^*}\Delta y$ cubic feet, and a weight of $62.5\pi\sqrt{y_i^*}\Delta y$ pounds. Since this slice is to be lifted $3 - y_i^*$ feet, the work done is

$$62.5\pi\sqrt{y_i^*}(3 - y_i^*)\Delta y$$

foot-pounds. Total approximate work done is

$$\sum_{i=1}^{N} 62.5\pi\sqrt{y_i^*}(3 - y_i^*)\Delta y$$

foot-pounds Exact work done is

$$W = \int_0^3 62.5\pi\sqrt{y}(3-y)\ dy$$

foot-pounds

Example 4, page 400:
The tank is the volume of revolution formed by the line $y = 2.5x$. The slice that extends from y_{i-1} to y_i can be approximated by a washer that has a thickness of Δy meters, a cross-sectional radius of approximately x_i^* meters corresponding to a value of $y_i^* \in [y_{i-1}, y_i]$, a cross-sectional area of approximately $\pi(x_i^*)^2 = \pi\frac{(y_i^*)^2}{6.25}$ square meters, a volume of approximately $\pi\frac{(y_i^*)^2}{6.25}\Delta y$ cubic meters, a mass of $1000\pi\frac{(y_i^*)^2}{6.25}\Delta y$ kilograms, and represents a force of $9800\pi\frac{(y_i^*)^2}{6.25}\Delta y$ newtons. Since this slice is to be lifted $10 - y_i^*$ feet, the work done is

$$9800\pi\frac{(y_i^*)^2}{6.25}(10 - y_i^*)\Delta y$$

foot-pounds. Total approximate work done is

$$\sum_{i=1}^{N} 9800\pi\frac{(y_i^*)^2}{6.25}(10 - y_i^*)\Delta y.$$

Since y_i^* varies in $[0, 8]$, exact work done is

$$W = \int_0^8 9800\pi\frac{y^2}{6.25}(10-y)\ dy$$

joules. We get the same answer as in the book.

Remark: We can use the same ideas for other shapes, so long as we can approximate the typical slice of volume.

Hydrostatic Pressure: Consider a rectangular container that has base area A square feet and depth d feet. When the containiner is filled with water, the pressure on the base of the container from the weight of the water above it is $P = F/A$. The weight of the water 62.5 pounds per cubic foot times Ad cubic feet, which yields $F = 62.5Ad$. The area of the base is A square feet. Hence the pressure is $\frac{62.5Ad}{A} = 62.5d$ pounds per square foot.

If the container has base area A square meters and depth d meters, then the mass of the water is Ad kg, the acceleration is 9.8 meters per square second, hence force of the water on the base is $F = 9.8Ad$ newtons, hence the pressure on the base is $\frac{9.8Ad}{A} = 9.8d$ newtons per square meter.

Hydrostatic pressure is regarded to be the same in all directions at any given depth.

A Hydrostatic Force Problem:
A metal plate in the shape bounded by the curves $y = x^2$ and $y = 18 - x^2$ (dimensions are feet) is submerged in water so that the top is 10 feet below water level. Calculate the total amout of hydrostatic force exerted by the water against the plate.

Solution: Since hydrostatic pressure (pounds of force per square foot of area) across a slice of the plate is the same, we will approximate the total force on a given slice of the plate by multiplying the area of the slice by the force per square foot, then add the result.

Segment each half of the plate into slices of width Δy. Assuming the approximate depth of the i^{th} slice is $20 - y_i^*$, the hydrostatic pressure on the plate at a depth of $20 - y_i^*$ is $62.5(20 - y_i^*)$ pounds per square foot. For $y_i^* \in [0, 9]$, the slice can be approximated by a rectangle with length extending between the corresponding x-coordinates on the curve $y = x^2$, namely $2\sqrt{y_i^*}$, for an area of $2\sqrt{y_i^*}\Delta y$ square feet and a force of

$$125(20 - y_i^*)\sqrt{y_i^*}\Delta y$$

pounds. For $y_i^* \in [9, 18]$, the slice can be approximated by a rectangle with length extending between the corresponding x-coordinates on the curve $y = 18 - x^2$, namely $2\sqrt{18 - y_i^*}$, for an area of $2\sqrt{18 - y_i^*}\Delta y$ square feet and a force of

$$125(20 - y_i^*)\sqrt{18 - y_i^*}\Delta y$$

pounds. Hence total force is

$$F = \int_0^9 125(20 - y)\sqrt{y}\ dy + \int_9^{18} (20 - y)\sqrt{18 - y}\ dy = 49,500$$

pounds. In the second integral, make the substitution $u = 18 - y$ to simplify the calculation.

Center of mass: Say objects with masses m_1 through m_k and total mass M are located at positions (x_1, y_1, z_1) through (x_k, y_k, z_k). The center of mass of the collection of objects is defined to be $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{m_1}{M}x_1 + \cdots + \frac{m_k}{M}x_k,$$

$$\overline{y} = \frac{m_1}{M}y_1 + \cdots + \frac{m_k}{M}y_k,$$

and

$$\overline{z} = \frac{m_1}{M}z_1 + \cdots + \frac{m_k}{M}z_k.$$

Given, for simplicity, a two-dimensional region with uniform mass density and bounded by the curves $y = f(x)$ and $y = g(x)$ over $[a, b]$, we approximate the region by rectangular slices and treat each rectangle as having mass equal to the area of the rectangle concentrated in the center of the rectangle. This yields

$$\overline{x} \approx \frac{f(\overline{x_1}) - g(\overline{x_1})\Delta x}{A}\overline{x_1} + \cdots + \frac{(f(\overline{x_n}) - g(\overline{x_n}))\Delta x}{A}\overline{x_n},$$

$$\overline{y} \approx \frac{(f(\overline{x_1}) - g(\overline{x_1}))\Delta x}{A}\frac{f(\overline{x_1}) + g(\overline{x_1})}{2} + \cdots + \frac{(f(\overline{x_n}) - g(\overline{x_n}))\Delta x}{A}\frac{f(\overline{x_n}) + g(\overline{x_n})}{2}.$$

Letting $n \to \infty$, we obtain

$$\overline{x} = \frac{\int_a^b x(f(x) - g(x))\ dx}{\int_a^b f(x) - g(x)\ dx},$$

$$\overline{y} = \frac{\int_a^b \frac{f(x)^2 - g(x)^2}{2}\ dx}{\int_a^b f(x) - g(x)\ dx}.$$

Note: If there is a mass density function ρ that varies with x-coordinate, we can modify the formulas above suitably.

Theorem of Pappus: Rotate plane figure about axis. Volume of revolution is area times distance centroid (center of mass assuming uniform density) travels.

Homework: Section 7.6, Problems 9, 11, 13, , 27, 43, 53