Eigenvalues and Eigenvectors Lecture

Let $\dim V = n$. Since $\dim \mathcal{L}(V) = n^2$, every $T \in \mathcal{L}(V)$ satisfies a polynomial equation $p(T) = 0$ of degree $\leq n^2$. (Later: we will improve this to degree $\leq n$). The minimal polynomial is defined to be the unique monic polynomial $p(x)$ of least degree such that $p(T) = 0$.

Eigenvalue and eigenvector of T: $Tv = \lambda v$ where $v \neq 0$. v and λ exist if and only if $T - \lambda I$ is not invertible.

Theorem: Let $p(x)$ be the minimal polynomial of T. Then λ is an eigenvalue if and only if it belongs to the base field and $p(\lambda) = 0$.

Proof: Let λ be an eigenvalue with eigenvector v. Then $0 = p(T)v = p(\lambda)v$, therefore $p(\lambda) = 0$. Conversely, if $p(\lambda) = 0$ then $p(x) = (x - \lambda)q(x)$, therefore $p(T) = (T - \lambda I)q(T)$. By minimality, $q(T) \neq 0$, so there is a vector v such that $q(T)v \neq 0$. This is an eigenvector of T corresponding to eigenvalue λ.

Corollary: Every complex operator on a finite-dimensional space has an eigenvalue.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + y, x + y)$. Matrix representation with respect to standard basis: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Minimal polynomial: $x^2 - 2x$. Eigenvalues: $\lambda = 0, 2$. To find eigenvectors, consider $T(T-2)e_1 = 0$ and $(T - 2)Te_2 = 0$.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x - y, x + y)$. Matrix representation with respect to standard basis: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Minimal polynomial: $x^2 - 2x + 2$. No eigenvalues or eigenvectors.

Example: Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(x, y) = (x - y, x + y)$. Matrix representation with respect to standard basis: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Minimal polynomial: $x^2 - 2x + 2$. Eigenvalues are $\lambda = 1 + 2i, 1 - 2i$. Eigenvectors: consider $T^2 - 2T + 2I$ applied to e_1 and e_2.

Example: Let $T : P(\mathbb{Z}_5) \to P(\mathbb{Z}_5)$ be defined by $Tp(x) = xp(x)$. Then $T^5 = T$, therefore $x^5 - x$ is a polynomial satisfied by T. Let $p(x)$ be the minimal polynomial. By the division algorithm, $x^5 - x = p(x)q(x) + r(x)$ for some remainder polynomial of degree less than the degree of the minimal polynomial. Since $r(T) = 0$, minimality forces $r(x) = 0$. Hence $p(x)$ is a divisor of $x^5 - x$. Note that $T^5 - T = T(T^4 - I) = T(T^2 - I)(T^2 + I) = T(T^2 - I)(T^2 - 4I) = (T - I)(T + I)(T - 2I)(T + 2I)$. Applying this in various permutations to x yields eigenvalues $0, 1, 2, 3, 4$. Hence $x^5 - x$ is a divisor of $p(x)$, so $p(x) = x^5 - x$.

Every linear operator on a finite-dimensional complex vector space has an upper-triangular matrix representation: by induction on dimension n. True for $n = 1$. More generally, let $T : V \to V$ be given. Write $V = Fu_1 \bigoplus W$ where $Tu_1 = \lambda u_1$. Let P project V onto W. Then PT maps W onto itself, and by the induction hypothesis PT has an upper-triangular matrix representation with respect to a basis $\{u_2, \ldots, u_n\}$. This implies that T has an upper-triangular matrix representation with respect to the basis $\{u_1, \ldots, u_n\}$.

Note that once T has been given an upper-triangular matrix representation U, its eigenvalues all appear along the diagonal. Reason: λ is an eigenvalue if and only if $T - \lambda I$ is not invertible. Using the basis which gives to the upper-triangular representation, $T - \lambda$ has matrix representation $U - \lambda I$, and this represents an invertible linear transformation if and only if one of the diagonal entries is zero.

Now consider real vector spaces. The same argument above implies that either T has a real eigenvalue or there exists a quadratic polynomial $a(T)$ such that $a(T)u = 0$ for some non-zero u (method: factor the minimal polynomial into linear and quadratic factors). If there are no eigenvalues then $U = \mathrm{span}_{\mathbb{R}}\{u, Tu\}$ has dimension 2 and T maps U into U. Summarizing, T has an invariant subspace of dimension ≤ 2.

Operators on real vector spaces have a block-upper-triangular matrix representation with blocks of size ≤ 2: by induction on dimension. True in dimension 1. More generally, find an invariant space U of dimension ≤ 2. Expand to $V = U \bigoplus W$. Let P be projection onto W and use the induction hypothesis to find a block-upper-triangular matrix representation of PT on W. This can be expanded to block-upper-triangular matrix representation of T. Note that this implies that operators on real vector spaces always have a real eigenvector, because at least one block has size 1.

Theorem: Let $\dim V = n$. For $T \in \mathcal{L}(V)$, the minimal polynomial of T has degree $\leq n$.

Proof: We will first do this under the assumption that V is a complex vector space. Let A be an upper-triangular matrix representation with respect to an appropriate basis. Let $\lambda_1, \ldots, \lambda_n$ be the entries on the diagonal. Then for each i, $T - \lambda_i I$ has matrix representation $A - \lambda_i I$, which is upper-triangular with a 0 in row i, column i. This implies that $(A - \lambda_1 I) \cdots (A - \lambda_n I) = 0$, which implies that the matrix representation of $(T - \lambda_1 I) \cdots (T - \lambda_n I)$ is 0, which implies $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$. Hence the minimal polynomial of T is a divisor of $(x - \lambda_1) \cdots (x - \lambda_n)$.

Now assume that V is a real vector space. Let A be the matrix representation of T. The entries of A are all real and $p(A) = 0$ for some complex polynomial $p(x)$ of degree $\leq n$. If $r(x)$ and $c(x)$ are the real polynomials derived from $p(x)$ by just using the real or the imaginary coefficients, we obtain $r(A) = 0$ and $c(A) = 0$. Hence A satisfies a non-zero real polynomial of degree $\leq n$, which implies that T does also.

Proving that the product of n $n \times n$ upper-triangular matrices, each with a diagonal zero entry in a different location, is zero: Introduce 2×2 block matrix multiplication. Product of upper triangular-matrices is upper-triangular. Using an induction argument, the product of the last $n-1$ matrices has 0 in the 2×2 position. Now multiply by the first matrix to see that the product is 0.