A Walk Through Combinatorics, Miklós Bóna

Chapters 3–4, 9 meetings: Counting Techniques and Binomial Identities

Counting concepts: permutations, multisets, multiset permutations, strings, bijection, subsets, permutations of n objects chosen k at a time, subsets of size k chosen from n elements, set complement, binomial coefficients, number of multisets of size k from an n-element set.

Counting technique concepts organized:

1. List, string, permutation, rearrangement.

2. Set, multiset, multiset permutation, subset.

3. Representation of objects and bijections. (a) Subsets correspond to binary strings. (b) Multisets correspond to non-negative integer solutions to $x_1 + x_2 + \cdots + x_k = n$. (c) Multisets correspond to strings of dots and bars. (d) Multisets of k items from $[n]$ correspond to weakly increasing sequences $1 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq n$, which correspond to strictly increasing sequences $1 \leq x_1 + 0 < x_2 + 1 < \cdots < x_k + (k-1) \leq n + k - 1$, which correspond to k-element subsets of $[n+k-1]$.

4. Decision trees. (a) Permutations of $[n]$: first element, second element, ... (b) Permutations of n objects chosen k at a time: first element, second element, ... (c) Permutations of n objects chosen k at a time: subset of k, permutation of k (d) All subsets of $[n]$: is 1 in, is 2 in, ... (e) Rearrangements of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$: subset of x_1 positions, subset of x_2 positions, ...

5. Formulas corresponding to decision trees: (a) $n!$ (b) $n(n-1)\cdots(n-k+1)$ (c) $\binom{n}{k}k!$, yielding formula for $\binom{n}{k}$ (d) 2^n (e) $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots = \frac{n!}{n_1!n_2!\cdots n_k!}$.

6. BOOKKEEPER problems: (a) all rearrangements (b) OO together: glueing technique (c) OO not present: set complement (d) Os separated by at least 1 letter: Os positions, rearrangements of what's left (e) Es separated by at 2 letters: E positions, rearrangements of what's left (f) consonants consecutive: where they are, how they are arranged, how the rest is arranged (g) consonants in alphabetical order: where they are, how they are arranged, how the rest is arranged (h) consonants all separated: where they are, how they are arranged, how the rest are arranged (i) first letter must be a vowel: must consider two cases and add the results (j) first letter must be a consonant: add cases or use set complement.

Exercises, page 49: (1) skip (2) map from odd to even: if it contains n, take it out. If it doesn't contain n, put it in. (3) Select 5 people, through out all male/all female committees. (4) If there are fewer than 49 flags, then some countries have same flag. How many possible flags? XYZ from RWBG is 24. If no flag used 3 times then each flag used at most 2 times, representing only 48 countries. Oops. (5) skip (6) skip (7) skip (8) set complement (9) glue (10) 5-subsets of [7] not containing both 6 and 7: set complement (11) $x_1+\cdots+x_5 = 7$ (12) First consider rearrangements of $1^5 2^5 3^5 4^5$, then toss out the bad ones. (13) If there are fewer than 60 2-course combinations, then the professor had two similar semesters. The number of 2-course combinations is $\binom{15}{2} = 105$. (14) Label the soup options 0-5, the main-course options 0-10, the dessert options 0-6, then choose one of each. (15) Decide where she is in the first 95 days. Rearrangements of $L^5 O^9 0$ beginning with L, every L followed by at least 6 Os. (16) skip (17) Count rearrangements of $1^{b_1} 2^{b_2} \cdots k^{b_k}$. (18) skip (19a) Choose soccer team, then choose basketball team from the rest (19b) choose soccer team, then choose basketball team (19c) add ways to choose two disjoint teams and ways to choose teams sharing one member. Sharing one member: choose the member, then round out the teams. (20) Total number of licence plates: 10^6. Number of these of this special type: choose the repeated digit, choose the other digits, form all rearrangements. Compare to 10% of the total. (21) skip (22) skip (23) First, decide which rows are occupied. Second, decide which columns are occupied. Toss out the other rows and columns and look at the resulting 8×8 board. Decide which column corresponds to each row. (24) skip

Supplementary exercises, page 53: (25) Count all 3-digit strings, toss out those starting with 0. Choose the digit appearing twice (10), choose the other digit (9), count the rearrangements (3). Toss out 0X0, 00X, 0XX. Total $10 \cdot 9 \cdot 3 - 9 - 9 - 9 = 243$. (26) Rearrangements formula yields $\frac{7!}{4!} = 210$. (27) Decide if 1 or 2 is in, then fill out the rest of the subset: $2 \cdot 2^{n-2} = 2^{n-1}$. (28) Toss out those that don't contain 1 or 2: $2^n - 2^{n-2} = 3 \cdot 2^{n-2}$. (29) Choose each digit: $4 \cdot 10 \cdot 5 = 200$. (30) Choose each digit: $9 \cdot 9 \cdot 8 \cdot 7 = 4536$. (31) Toss out those that omit 1. All not starting with 0: $9 \cdot 10 \cdot 10 \cdot 10 = 9000$. Those not starting with 0 and not containing 1: $8 \cdot 9 \cdot 9 \cdot 9 = 5832$. Difference is 3168. (32) If we don't worry out leading 0s, we can calculate e_n = number of n-digit strings in 0-9 with an even sum and o_n = number of n-digit strings in 0-9 with an odd sum using a recurrence relation. Then we want to calculate e_3 and toss out strings beginning with 0, which leaves $e_3 - e_2 = \frac{1000}{2} - \frac{100}{2} = 450$.

(33a) First choose a subset of 2 positions for 1/2, then 2 remaining positions for 3/4, then permute the remaining digits: $\binom{n}{2}\binom{n-2}{2}(n-4)! = \frac{n!}{4}$. (33b) Choose a subset of 3 positions for 1, 2, 3, then decide on the relative order of 2/3, then permute the rest: $\binom{n}{3} \cdot 2 \cdot (n-3)! = \frac{n!}{3}$. (34) If n is odd the answer is 0. If $n = 2k$ then consider k consecutive pairs of positions. Choose one of each pair to hold the even number. Then permute the even numbers in these positions, then permute the odd numbers in the remaining positions. $2^k \cdot k! \cdot k! = 2^k(k!)^2$. (35) Choose each exponent in the given range: $(a_1+1)(a_2+1)\cdots(a_k+1)$. (36) skip (37) skip (38) Let the couples be numbers 1 through n. Assign to each couple a random number in the range 0,1,2. 0 indicates nobody is chosen, 1/2 indicates who is chosen. 3^n. (39) skip (40) Given A, the number of pairs (A, B) can be obtained from all (A, X) by subtracting those X disjoint from A. Now sum over all A. Alternatively, assign to each number in $[n]$ the symbol a, b, ab, x. If a number is assigned a, place in A. If a number is assigned b, place in B. If a number is assigned ab, place in both sets (the intersection). If a number is assigned x, don't place it in either set. Total number of strings is 4^n. Bad strings are those that don't contain an ab symbol, of which there are 3^n. Total is $4^n - 3^n$. (41) Use symbols ac, bc, abc, c, x, toss out strings not containing abc symbol. Total is $5^n - 4^n$. (42) Assign each person a number indicating what that person did (gave a talk or not, talk selected or not). There are 5 categories, and we must avoid sequences where 2 categories are absent. Hence $5^n - 3^n$. (43) Assign students to faculty. $13 \cdot 12 \cdot 11 \cdot 10 = 17160$. (44) skip (45) Rearrangements of 444455 or 444446. Adding, obtain $\frac{6!}{4!2!} + \frac{6!}{5!} = 21$. (46) The probability that a string of length 4 in 1–6 will not show a 6 is $\frac{5^4}{6^4} = 48.2253\%$. Brenda has better odds. (47) In other words, what is the average smallest element in a subset of size k chosen from $[n]$. Number of subsets of size k beginning with a is number of subsets of size $k-1$ chosen from $[n-a]$, namely $\binom{n-a}{k-1}$. So the average is $\sum_{a=1}^{n-k+1} a\binom{n-a}{k-1}/\binom{n}{k}$. Using Mathematica this seems to simplify to $\frac{n+1}{k+1}$. To prove this one can attempt induction on k or try to simplify $\sum_{a=1}^{n-k+1} a\binom{n-a}{k-1}$ using generating functions. Note that $\sum_{a=0}^{\infty} ax^a = \frac{x}{(1-x)^2}$ and $\sum_{a=0}^{\infty}\binom{a}{b}x^a = \frac{x^b}{(1-x)^{b+1}}$, therefore $\sum_{a=0}^{n-b} a\binom{n-a}{b} = [x^n]\frac{x^{b+1}}{(1-x)^{b+3}} = [x^{n+1}]\frac{x^{b+2}}{(1-x)^{b+3}} = \binom{n+1}{b+2}$. Setting $b = k-1$ we obtain $\binom{n+1}{k+1}$. So the average is $\binom{n+1}{k+1}/\binom{n}{k} = \frac{n}{k+1}$. (48) Choosing one column for each row we obtain $n!$. (49) Sophomore 1 picks a junior and a senior in n^2 ways. Sophomore 2 picks a junior and senior in $(n-1)^2$ ways, etc: $(n!)^2$.

(50) First choose 16 teams to place in the American Conference, thereby determining the teams in the National Conference: $\binom{32}{16}$. Number of ways to form a sequence of four divisions: $\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{16!}{(4!)^2}$. Final result is $\binom{32}{16}\frac{16!}{(4!)^2}^2 = 2390461829733887910000000$. (51) We must replace $\binom{32}{16}$ by $2\binom{30}{15}$, which changes the final result to $2\binom{30}{15}\frac{16!}{(4!)^2}^2 = 1233786750830393760000000$.

Binomial identities concepts: counting things in two different ways, binomial theorem, binomial theorem evaluations, Pascal's triangle and identity, assigning a combinatorial interpretation to a summation formula, simplifying a sum using a combinatorial interpretation, binomial coefficient inequalities (skip), multinomial theorem, generalized binomial theorem (defer until generating functions).

Binomial identities concepts organized:

1. Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: Multiplying $(x+y)^n$ out we obtain every possible string of length n in x and y. The number of these that contain k x's and $n-k$ y's is equal to $\binom{n}{k}$.

2. Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{e_1+e_2+\cdots+e_k=n} \frac{n!}{e_1!e_2!\cdots e_k!} x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$$

Proof: Multiplying $(x_1 + x_2 + \cdots + x_k)^n$ out we obtain every possible string of length n in x_1 through x_k. The number of these that contain e_1 x_1's, ..., e_k x_k's is $\frac{n!}{e_1!e_2!\cdots e_k!}$.

3. Binomial identities via evaluation: choose numerical values for x and y. Or: use differentiation and integration. To prove a formula, figure out what operations to perform on the binomial theorem identity. This yields proofs of Theorem 4.2 (p. 68), Theorem 4.4 (p. 68), Theorem 4.6 (p. 70).

4. Combinatorial proofs: one can take a known formula and supply a proof that does not involve binomial evaluation or algebra. To prove Theorem

4.2, prove that there are an equal number of even and odd subsets. To prove Theorem 4.4, count all subsets of $[n]$ using multiplication principle (LHS) and counting subsets of each size (RHS). To prove Theorem 4.6, count all subsets that have an element circled by choosing the circled element first (RHS), then count these subsets by choosing the circled element second (LHS).

Identities inspired by combinatorial proofs: why not count all subsets that have two different circled elements. Choosing the elements to circle first yields $\binom{n}{2}2^{n-2}$. Choosing the elements to circle second yields $\sum_{k=2}^{n}\binom{n}{k}\binom{k}{2}$.

c circles: $\binom{n}{c}2^{n-c}=\sum_{k=c}^{n}\binom{n}{k}\binom{k}{c}$.

To prove theorem 4.7 (p. 71), count subsets of $[n+m]$ of size k directly (LHS) or by deciding how many elements fall in $[n]$ (RHS).

A variation on this: count k-subsets of $[n+m]$ with an element in $[n]$ circled. Choosing the circle first: $n\binom{n+m-1}{n-1}$. Choosing the circle second: $\sum_{i=1}^{n} i\binom{n}{i}\binom{m}{k-i}$.

c circles: $\binom{n}{c}\binom{n+m-c}{k-c}=\sum_{i=c}^{n}\binom{i}{c}\binom{n}{i}\binom{m}{k-i}$.

a circles from $[n]$, b circles from $\{n+1,\dots,n+m\}$: $\binom{n}{a}\binom{m}{b}\binom{n+m-a-b}{k-a-b}=\sum_{i=a}^{n}\binom{i}{a}\binom{k-i}{b}\binom{n}{i}\binom{m}{k-i}$.

Pascal's triangle: rows 0, 1, 2, ..., n. In row n, columns 0, 1, ..., n. If we left-justify, we see evidence for Pascal's identity, Theorem 4.3 (p. 68). Proving it: count $(k+1)$-subsets of $[n+1]$ directly (RHS) and by organizing according to presence or absence of the number $n+1$ (LHS).

Generalizing this: we can either use this as a recurrence relation to expand it out, or we can count $(k+1)$-subsets of $[n+1]$ according to largest element. This yields Theorem 4.5 (p. 69).

5. Another combinatorial identity: non-negative integer solutions to $x_1+\cdots+x_k=n$ organized in various ways. For example, organized by value of x_1: directly yields n dots, $k-1$ bars, $\binom{n+k-1}{k-1}$. But when $x_1=i$ we have $n-i$ dots and $k-2$ bars, which yields $\binom{n-i+k-2}{k-2}$. Hence

$$\binom{n+k-1}{k-1}=\sum_{i=0}^{n}\binom{n-i+k-2}{k-2}.$$

Equivalently,

$$\binom{N+1}{K+1}=\sum_{i=0}^{N-K}\binom{N-i}{K}.$$

Number of n dot, k bar diagrams where one of the dots before the first bar is circled: interpreting the circled dot as a bar we obtain an $n-1$ dot, $k+1$ bar diagram, hence $\binom{n+k}{k+1}$. But when $x_1 = i$ we have i ways to choose the circled dot and $n-i$ dots and $k-1$ bars left over, which yields $i\binom{n-i+k-1}{k-1}$. Hence

$$\binom{n+k}{k+1} = \sum_{i=1}^{n} i\binom{n-i+k-1}{k-1}.$$

Equivalently,

$$\binom{N+1}{K+2} = \sum_{i=1}^{N-K} i\binom{N-i}{K}.$$

Challenge: obtain a binomial identity by counting n-dot, k-bar diagrams where c of the dots before the first bar are circled.

Exercises, p. 76: (1) skip (2) skip (3) Subsets of $[n]$ with one element circled and another element squared (4) Choose m elements from $[n]$, then circle k of these (LHS), choose the k circled elements first, then round out the subset (RHS). (5) skip (6) skip (7) skip (8) The sum of the binomial coefficients is $2^{2n} = 4^n$. (9) skip (10) skip (11) skip (12) skip (13) skip (14) skip (15) skip (16) skip (17) Multinomial Theorem evaluation with variables equal to 1 (18) Multinomial Theorem evaluation with variables $1, -1, 1$ (19) skip (20) skip (21) skip (22) skip (23) skip (24) skip (25) Derivative of geometric series (26) skip (27) skip.

Supplementary Exercises, p. 79: (28) skip (29) skip (30) skip (31) skip (32) We are counting sequences of Xs and Ys where X means go right and Y means go up. These contain n Xs and k Ys, hence $\binom{n+k}{n}$. (33) skip (34) skip (35) skip (36) skip (37) skip (38) Special case of Theorem 4.7 (p. 71) with $\binom{n}{n-i}$ replaced by $\binom{n}{i}$. (39) Same as (38) but now introducing a circled dot where needed. (40) Binomial Theorem evaluation. (41) skip (42) Taking the definite integral between 0 and t yields $\frac{(1+t)^{n+1}}{n+1} - \frac{1}{n+1} = \sum_{k=0}^{n} \binom{n}{k}\frac{t^{k+1}}{k+1}$. (43) Use $t = -1$ in (42). (44) See (42). (45) Add the Binomial Theorem evaluation with $(x, y) = (2, 1)$ to the Binomial Theorem evaluation with $(x, y) = (-2, 1)$ and cancel out terms with opposite sign. (46) Same idea as in (45). (47) Expand $(xi + y)^n + (-xi + y)^n$ using the Binomial Theorem, where i represents the complex number $\sqrt{-1}$. Cancel out terms with opposite sign. Now choose x and y appropriately. (48) – (60) skip.

Additional problems:

61. Evaluate $\sum_{k=1}^{n} k\binom{n}{k}3^k$.

62. Evaluate $\sum_{k=1}^{n} k^2\binom{n}{k}4^k$.

63. Evaluate $\sum_{k=1}^{n} \binom{n}{k}\frac{5^k}{(k+1)(k+2)}$.

64. Find the binomial identity that can be obtained by counting subsets of $[n]$ that contain a red number, a blue number, a green number (all other numbers colored black).

65. Find the binomial identity that can be obtained by counting subsets of $[n]$ that contain a circled number and a squared number, allowing the possibility that a number can be both circled and squared at the same time as long as the square goes around the circle).

66. Find the binomial identity that can be obtained by counting dots and bars diagrams with n dots and k bars where two of the dots before the first bar are circled.

Chapters 5 and 7, 9 meetings: Partitions, Inclusion-Exclusion

Concepts: Compositions, weak compositions, set partitions, Stirling numbers of the second kind, surjective functions, Bell numbers, integer partitions, Ferrers diagram, restricted integer partitions, identities involving set partitions and restricted integer partitions.

Concepts organized: A partition of a collection of objects is a way of organizing them into parts. A general model is distributing balls into boxes. The balls can be identical or distinct (numbered or given different names). The boxes can be identical or distinct. Moreover, the boxes can be wide or narrow. When the balls are distinct, the order of the balls inside a wide box doesn't matter, but when the order of the balls inside a narrow box (like a Pringle's container or a test tube) does matter.

1. Compositions. Consider organizing n identical balls into k distinct boxes. Since the balls are identical, we disregard the order of balls in a box and just keep track of number of balls in each box. So we are counting solutions to $x_1 + \cdots + x_k = n$. When each $x_i \geq 1$ these are called compositions (non-empty boxes). When each $x_i \geq 0$ these are called weak compositions

(empty boxes allowed). If $comp(n,k)$ represents number of compositions of n identical balls into k distinct boxes, then by keeping track of the number of balls in the first box we obtain

$$comp(n,k) = \sum_{i=1}^{n-k+1} comp(n-i, k-1).$$

Of course, we know that $comp(n,k)$ can be represented by diagrams with $n-k$ dots and $k-1$ bars (after adding a dot before each bar and after the last bar), hence $comp(n,k) = \binom{n-1}{k-1}$. Making the substitution we obtain

$$\binom{n-1}{k-1} = \sum_{i=1}^{n-k+1} \binom{n-i+k-2}{k-2}.$$

2. Set partitions. $S(n,k)$ is the number of ways n distinct balls can be distributed into k identical boxes where each box contains at least one ball and the order of the balls in the boxes is not relevant and the arrangement of the boxes is not relevant. In other words, all sets of sets $A_1, \ldots, A_k$ where each set is non-empty, no two sets intersect, and their union is $\{1, 2, \ldots, n\}$. It is not easy to find a formula for $S(n,k)$, but we can calculate the numbers using a recurrence relation. Clearly we must have $n \geq k$ as in the binomial coefficients, so we can arrange all these numbers as in Pascal's Triangle. We have $S(1,1) = 1$. Having computed rows 1 through $n-1$ in the triangle, we can calculate $S(n,k)$ as follows: If n is in a box by itself, there are $S(n-1,k-1)$ ways to distribute the remaining balls. If n is not in a box by itself there are $S(n-1,k)$ ways to distribute the remaining balls and k ways to decide which set of these n belongs to. So we obtain $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

3. Surjections. A surjection $f : A \to B$ is any function that maps onto B. When $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, \ldots, k\}$ we can think of f as distributing n distinct balls into k distinct boxes with no box empty. Analogous to the way we computed $\binom{n}{k}$, we can count surjections by a two step decision process: first count set partitions of $[n]$ into k non-empty disjoint identical sets, then arrange them into every possible order. The number of surjections is $k!S(n,k)$. Since we can generate $S(n,k)$ algorithmically, we can generate the number of surjections algorithmically.

4. Counting all functions via surjections: Every function $f : A \to B$ can be regarded as a surjection onto some non-empty subset of B. When $A = [n]$ and

$B = [k]$ there are k^n functions, since there are k choices for each $f(i)$. We can organize functions into categories, where category i functions are surjective onto a subset of size i in $[k]$. To count these, first choose the i elements, then choose a surjection. Total: $\binom{k}{i} S(n,i)$. Adding up the categories,

$$k^n = \sum_{i=1}^{k} \binom{k}{i} i! S(n,i) = \sum_{i=1}^{k} k(k-1)\cdots(k-i+1)S(n,i).$$

Now if we form the polynomial

$$x^n - \sum_{i=1}^{k} x(x-1)\cdots(x-i+1)S(n,i),$$

we have shown that it has a root at k for $k = 1, 2, 3, \ldots$. Hence the polynomial is zero, because non-zero polynomials have a finite number of roots. This implies that

$$x^n = \sum_{i=1}^{k} x(x-1)\cdots(x-i+1)S(n,i)$$

for any arbitrary numerical value for x.

5. Bell numbers. $B(n)$ counts all the ways to partition n distinct balls into any arbitrary number of identical non-empty boxes. To obtain a recurrence relation for $B(n)$, we have $B(1) = 1$ and, for $n \geq 1$, first decide which i balls accompany $n+1$ in $\binom{n}{i}$ ways, then partition the remaining $n-i$ balls: $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(n-i) = \sum_{i=0}^{n} \binom{n}{n-i} B(n-i) = \sum_{i=0}^{n} \binom{n}{i} B(i)$ assuming $B(0) = 1$.

6. Integer partitions of n. These are weakly descending sequences $a_1 \geq a_2 \geq \cdots$ of positive integers whose sum is n. $p(n)$ is the number of partitions of n. People who have looked at partitions have found that there are many subcategories of partitions of equal size. Examples:

a. $p(n, \text{odd}) = p(n, \textit{distinct})$.

b. $p(n, \text{no part divisible by 3}) = p(n, \text{no part appears more than 2 times})$.

c. $p(n, k \text{ parts}) = p(n, \text{largest part } k)$.

Many partition identities can be proved by manipulating Ferrers diagrams. Represent $a_1 + a_2 + \cdots$ by rows of the corresponding number of squares, left-justified. The conjugate partition is obtained from counting the squares in each column. We obtain a bijection from partitions to partitions by forming the conjugate. Under this bijection, partitions with k parts get mapped to partitions with largest part k.

Self-conjugate partitions are those that are equal to their own conjugate. These are in one-to-one correspondence with partitions with distinct odd parts: decompose odd numbers into L shapes.

Counting partitions that have all parts ≥ 2: bad partitions have smallest part 1. There are $p(n-1)$ of these. So partitions of n with all parts of size ≥ 2 are counted by $p(n) - p(n-1)$.

7. Number of set partitions of $[n]$ of type $a_1 + a_2 + \cdots + a_k$: we seek partitions where the numbers record the different occupancies. The occupancies are listed in descending order. Let's write the integer partition equivalently in the form $1^{m_1} 2^{m_2} \cdots n^{m_n}$, meaning there are i boxes of size i. First we will count set partitions of type i^{m_i} by counting ordered set partitions (boxes have identities): choose i balls for box 1 in $\binom{n}{i}$ ways, then choose i balls for box 2 in $\binom{n-i}{i}$ ways, etc. This yields $\frac{n!}{i! \cdots i!} = \frac{n!}{(i!)^{m_i}}$. A second way to count these: first choose an unordered set partition, then choose an identity for each box: $Xm_i!$. Given $\frac{n!}{(i!)^{m_i}} = Xm_i!$, there are $\frac{n!}{(i!)^{m_i} m_i!}$ unordered set partitions of type i^{m_i}. More generally, to generate unordered set partitions of type $1^{m_1} 2^{m_2} \cdots n^{m_n}$, choose m_1 balls for use in a set partition of type 1^{m_1}, then choose $2m_2$ balls for use in a set partition of type 2^{m_2}, and so forth, then choose sequence of unordered set partitions. This yields

$$\frac{n!}{m_1!(2m_2)! \cdots (nm_n)!} \times \frac{m_1!}{(1!)^{m_1} m_1!} \times \frac{(2m_2)!}{(2!)^{m_2} m_2!} \times \cdots \times \frac{(nm_n)!}{(n!)^{m_n} m_n!} =$$

$$\frac{n!}{(1!)^{m_1} (2!)^{m_2} \cdots (n!)^{m_n} m_1! m_2! \cdots m_n!}.$$

Exercises, p. 105: (1) Using the recurrence relation,

$$S(n,3) = S(n-1,2) + 3S(n-1,3).$$

To count $S(n,2)$, choose a subset of $[n-1]$ of size $\leq n-1$ to accompany n: $2^{n-1} - 1$. So we have

$$S(n,3) = 2^{n-2} - 1 + 3S(n-1,3).$$

We will guess a solution of the form $S(n,3) = a2^n + b3^n + c$. Plugging in $n = 3,4,5$ yields $a = -\frac{1}{2}$, $b = \frac{1}{6}$, $c = \frac{1}{2}$. This yields

$$S(n,3) = \frac{-3 \cdot 2^n + 3^n + 3}{2}.$$

(2) skip (3) skip (4) (skip) (5) Iterate the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

We obtain

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) =$$
$$S(n-1,k-1) + kS(n-2,k-1) + k^2S(n-2,k) =$$
$$S(n-1,k-1) + kS(n-2,k-1) + k^2S(n-3,k-1) + k^2S(n-3,k) = \cdots$$

Keep on going until the last two terms are $k^{n-1}S(0,k-1) + k^nS(0,k)$. Dropping the zero summands, we obtain

$$S(n,k) = \sum_{a=1}^{n-k+1} k^{a-1}S(n-a,k-1).$$

(6) See notes above. (7) $p(n+k, k \text{ parts}) = p(n+k, \text{largest part } k) = p(n, \text{largest part} \leq k)$, the last equality by stripping off the part of size k. (8) skip (9) skip (10) skip (11) $p(n) - p(n-1) = p(n, \text{smallest part } 2) = p(n, \text{first 2 parts equal})$, the last equality by looking at the conjugate partitions. (12a) Looking at the conjugate partition, the three smallest parts are 1, so ignoring them we obtain $p(n-3)$. (12b) After stripping off the three smallest parts of 1 and looking at the conjugate partition, we see partitions of $n-3$ with the first 2 parts equal. By (11) this is $p(n-3) - p(n-4)$. (13) The conjugate partitions have smallest part of size 3. To obtain these, subtract from $p(n)$ the partitions of n with smallest part 2 and with smallest part 1. Smallest part 1: $p(n-1)$. Smallest part 2: Removing the last 2 we obtain partitions of $n-2$. We want to discard any that have smallest part 1. So we get $p(n-2) - p(n-3)$. Altogether $p(n) - p(n-1) - p(n-2) + p(n-3)$. (14) skip (15) skip (16) skip.

Supplementary Exercises, p. 106: (17) Composition of 15: a solution to $x_1 + \cdots + x_k = 15$ where each $x_i \geq 1$ and $k \geq 1$. We proved above that $comp(15,k) = \binom{14}{k-1}$. Summing over all k yields 2^{14}. We want to discard compositions that start with 1. These correspond to compositions of 14, of which there are 2^{13}. So we obtain $2^{14} - 2^{13} = 2^{13} = 8192$. (18) Set partitions oxf $[10]$ with $\{1\}$ as a part correspond to set partitions of $\{2, \ldots, 10\}$, which are counted by $B(9)$. So we want $B(10) - B(9) = 94828$, using the recurrence relation to compute Bell numbers. (19) All partitions of $[8]$ into two sets: $S(8, 2) = 127$. Partitions of $[8]$ of type $4 + 4$: $\frac{8!}{(4!)^2 \cdot 2!} = 35$. Subtracting, obtain 92. (20) skip (21) Divide each even part by 2. (22) skip (23) Dividing $x_1 + \cdots + x_k = 10$ by 2 we see a composition of 5. There are $2^4 = 16$ such compositions. (24) Subtracting 1 from each part we see solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ with non-negative even solutions. Dividing by 2 we see solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 10$ with non-negative solutions. 10 dots and 4 bars yields $\binom{14}{4} = 1001$ solutions. (25) skip (26) Did this already in Problem (1). (27) At least 1 and at most 2 sets in the partition can have more than one ball in it. Exactly 1: choose 3 balls to place in the same set, then put the remaining balls in singleton sets. Exactly 2: Choose 4 balls, create a set partition of type $2 + 2$ using these, then place the rest of the balls in singleton sets. Hence

$$S(n, n-2) = \binom{n}{3} + \binom{n}{4} \frac{4!}{(2!)^2 \cdot 2} = \frac{3n^4 - 14n^3 + 21n^2 - 10n}{24} =$$

$$\frac{(n-2)(n-1)n(3n-5)}{24}.$$

(28) At least 1 and at most 3 sets in the partition can have more than one ball in it. Exactly 1: choose 4 balls to place in the same set, then put the remaining balls in singleton sets. Exactly 2: choose 5 balls, create a set partition of type $2+3$ using these, then place the rest of the balls in singleton sets. Exactly 3: choose 6 balls, create a set partition of type $2 + 2 + 2$ using these, the place the rest of the balls in singleton sets. Total:

$$S(n, n-3) = \binom{n}{4} + \binom{n}{5} \frac{5!}{2!3!} + \binom{n}{6} \frac{6!}{(2!)^3 3!} =$$

$$\frac{n^6 - 11n^5 + 47n^4 - 97n^3 + 96n^2 - 36n}{48} = \frac{(n-3)^2(n-2)^2(n-1)n}{48}.$$

(29) skip (30) skip (31) skip (32) Given an arbitrary set partition of $[n]$, merge all the singleton blocks together and toss in $n + 1$ to create a set partition of $[n+1]$ in which the only singleton block (potentially) is $\{n+1\}$.

This implies $B(n) = F(n+1) + F(n)$, since set partitions of $[n+1]$ with no singleton blocks are counted by $F(n+1)$ and set partitions of $[n+1]$ with only $\{n+1\}$ as a singleton block are in 1:1 correspondence with set partitions of $[n]$ with no singleton blocks, and the latter are counted by $F(n)$. (33) We can organize set partitions of $[n+1]$ with no singleton blocks according to how many elements accompany $n+1$. For $i = 1$ to n, choose i elements to accompany $n+1$, then form a set partition of the remaining $n-i$ elements with no singleton blocks. This yields

$$F(n+1) = \sum_{i=1}^{n} \binom{n}{i} F(n-i) = \sum_{i=1}^{n} \binom{n}{n-i} F(n-i) = \sum_{i=0}^{n-1} \binom{n}{i} F(i).$$

This requires $F(0) = 1$. (34) skip (35) Let A_n be the set of all compositions of n with parts of size ≥ 2. We can decompose this into the union of B_n and C_n, where B_n contains compositions in A_n with first part 2 and B_n contains compositions in A_n with first part ≥ 3. We obtain a bijection between B_n and A_{n-2} by dropping the first part of each composition in B_n. We obtain a bijection between C_n and A_{n-1} by subtracting 1 from the first part of each composition in C_n. Therefore $|A_n| = |A_{n-2}| + |A_{n-1}|$, which implies $a_n = a_{n-2} + a_{n-1}$. (36) Let A'_n be the set of all compositions of n with parts of size ≥ 3. We can decompose this into the union of B'_n and C'_n, where B'_n contains compositions in A'_n with first part 3 and C'_n contains compositions in A'_n with first part ≥ 4. We obtain a bijection between B'_n and A'_{n-3} by dropping the first part of each composition in B'_n. We obtain a bijection between C'_n and A'_{n-1} by subtracting 1 from the first part of each composition in C'_n. Therefore $|A'_n| = |A'_{n-3}| + |A'_{n-1}|$, which implies $b_n = b_{n-3} + b_{n-1}$.

Inclusion-Exclusion concepts: inclusion-exclusion formula, derangements, set partitions

1. Union of sets.

2. Intersection of sets.

3. Permutations containing a fixed point: union of sets.

4. Derangements: intersection. Can be expressed as universe minus union.
5. Compositions with upper limits: intersection (upper limits imposed). Can be expressed as universe minus union.

6. Ordered set partitions of $[n]$ into k non-empty sets: intersection (each set non-empty). (Think of ways to load n people onto k numbered buses, no

bus empty.) Can be expressed as universe minus union. To obtained $S(n,k)$, divide by $k!$.

7. Let $A_1, \dots A_n$ be sets. Name the elements in $A_1 \cup \cdots \cup A_n$ 1 through u. We wish to calculate u, the number of elements in the union. Let M_k be the multiset obtained by tossing all the k-fold intersections of sets together, with multplicity. Let $i \in [u]$ be given. How many times does i appear in M_k? Say that i belongs to exactly $s(i)$ of the sets $A_1, \dots, A_n$. Is $s(i) < k$ then i doesn't appear at all, so 0 times. If $s(i) \geq k$, i appears as many times as one can choose k of these sets and form an intersection. So the answer is always $\binom{s(i)}{k}$. This yields

$$|M_k| = \sum_{i=1}^{u} \binom{s(i)}{k}.$$

We will prove that

$$u = \sum_{k=1}^{n} (-1)^{k-1} |M_k|.$$

We have

$$\sum_{k=1}^{n} (-1)^{k-1} |M_k| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{i=1}^{u} \binom{s(i)}{k}.$$

The sum on the right-hand side can be reorganized into

$$\sum_{i=1}^{u} \sum_{k=1}^{n} (-1)^{k-1} \binom{s(i)}{k}.$$

Each of the expressions $\sum_{k=1}^{n} (-1)^{k-1} \binom{s(i)}{k}$ is equal to 1 by the Binomial Theorem. Hence we obtain

$$\sum_{k=1}^{n} (-1)^{k-1} |M_k| = \sum_{i=1}^{u} 1 = u.$$

Exercises, p. 142: (1) skip (2) skip (3) We are counting positive integers ≤ 210 that have no common divisor with 210. Since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, we want to discard numbers that are divisible by 2 or 3 or 5 or 7. We can count the numbers we are discarding by a union. (4) Discard numbers that are divisible by p or q or r. (5) Discard numbers that are divisible by p_1 or p_2 or ... or p_k. (6) skip (7) skip (8) Do an example, then derive the formula.

In general, if n is divisible by the primes p_1 through p_k, we want to discard numbers divisible by one of these. (9) skip (10) We are seeking permutations where $p_1 > p_2$ and $p_4 > p_5$ and $p_6 > p_7$. So we must discard permutations where $p_1 < p_2$ or $p_4 < p_5$ or $p_6 < p_7$. Label the sets A_1, A_4, A_7. Size of A_i: choose a subset of 2 elements to place in positions i and $i+1$, then permute the remaining elements: $\binom{8}{2}6!$. Size of $A_i \cap A_j$: choose a subset of 2 elements to place in positions i and $i+1$, choose a subset of 2 elements to place in positions j and $j+1$, then permute the remaining elements: $\binom{8}{2}\binom{6}{2}4!$. Size of $A_i \cap A_j \cap A_k$: $\binom{8}{2}\binom{6}{2}\binom{4}{2}2!$. (11) $A_4 \cap A_5$: 3 positions determined, and the elements must be in descending order, so we choose a subset of 3. (12)-(14) skip

Supplementary Exercises, p. 143: (15) A_1 = set of partitions containing $\{1\}$, A_2 = set of partitions containing $\{n\}$, $A_1 \cap A_2$ = set of partitions containing both $\{1\}$ and $\{n\}$, $|A_1 \cup A_1| = |A_1| + |A_2| - |A_1 \cap A_2| = 2B(n-1) - B(n-2)$. (16) A_i = set of permutations containing (i), $A_i \cap A_j \cap \cdots \cap A_k$ = set of permutations containing $(i)(j)\cdots(k)$, hence $|A_1 \cup A_2 \cup A_3| = n_1 - n_2 + n_3 = 3(n-1)! - 3(n-2)! + (n-3)!$ (17) skip (18) A_1 = set of permutations containing (12), A_2 = set of permutations containing (34), $A_1 \cap A_2$ = set of permutations containing (12)(34), $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 2(n-2)! - (n-4)!$ (19) First consider $n = 2k$ where $k \geq 1$. Bad permutations are in $A_1 \cup A_2$ where A_1 = set of permutations where $p_1 \in \{2, 4, \ldots, 2k\}$ and A_2 = set of permutations where $p_n \in \{2, 4, \ldots, 2k\}$. Good permutations = all minus bad $= (2k)! - |A_1| - |A_2| + |A_1 \cap A_2| = (2k)! - k(2k-1)! - k(2k-1)! + k(k-1)(2k-2)! = (2k)! - (2k)(2k-1)! + k(k-1)(2k-2)! = k(k-1)(2k-2)!$. Second, consider $n = 2k+1$ where $k \geq 1$. Bad permutations are in $A_1 \cup A_2$ where A_1 = set of permutations where $p_1 \in \{2, 4, \ldots, 2k\}$ and A_2 = set of permutations where $p_n \in \{2, 4, \ldots, 2k\}$. Good permutations = all minus bad $= (2k+1)! - |A_1| - |A_2| + |A_1 \cap A_2| = (2k+1)! - k(2k)! - k(2k)! + k(k-1)(2k-1)! = (2k+1)! - 2k(2k)! + k(k-1)(2k-1)!$. (20) skip (21) First choose the identity of the n-cycle, then choose a derangement of the remaining elements: $nD(n-1)$. (22) skip (23) This is obtained from $S(n,k)$ by subtracting set partitions of $[n]$ where part 1 is singleton or part 2 is singleton or ... . Setting A_i = set partitions of $[n]$ into k parts where part i is singleton, the intersection of any j of these is a set partition of $[n]$ into k parts where the j corresponding parts are singleton, and the number of these is $S(n-j,k-j)$. So in all we have

$$F_k(n) = S(n,k) - \sum_{j=1}^{k} (-1)^{j-1} \binom{n}{j} S(n-j,k-j) = \sum_{j=0}^{k} (-1)^{j} \binom{n}{j} S(n-j,k-j).$$

(24) skip (24) skip (26) All minus rearrangements where 11 appears or where 22 appears. All: $\frac{7!}{2!2!}$. Bad: $A_1 + A_2 - A_{12} = \frac{6!}{2!} + \frac{6!}{2!} - 5!$. Total = all - bad = 660. (27) – (31) skip (32) $F(n) = B(n)-$ number of set partitions of $[n]$ containing at least one singleton blocks. To count the latter, let A_i be the set of set partitions of $[n]$ with block $\{i\}$. Then the intersection of k of these sets contain set partitions with k prescribed blocks, and the rest of the elements of $[n]$ fall into set partitions of $n-k$ elements. This yields $N_k = \binom{n}{k} B(n-k)$, hence $F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} B(n-k)$. (33)–(35) skip (36) A permutation of $[n]$ has anywhere from 0 to n fixed points. To generate the typical permutation with k fixed points, first choose a subset of k elements out of $[n]$ to be fixed points, then derange the remaining $n-k$ elements. The number of these is $\binom{n}{k} D(n-k)$. Counting permutations by how many fixed points they contain we obtain

$$n! = \sum_{k=0}^{n} \binom{n}{k} D(n-k) = \sum_{k=0}^{n} \binom{n}{n-k} D(n-k) = \sum_{i=0}^{n} \binom{n}{i} D(i).$$

Chapter 8, 11 meetings: Generating Functions

Generating function concepts: sequence of numbers, recurrence relation, ordinary generating function, closed (or explicit) formula for a generating function, geometric series, partial fraction decomposition, coefficient extraction, products of generating functions, substitutions, differentiation, partition generating functions, partition identities via generating functions, planar binary trees, powers of a generating function with zero constant term, composition of two generating functions (both with constant term 0), exponential generating function, product of exponential functions, the exponential formula, exponential composition.

Generating function concepts organized:

1. Formal power series: geometric series, powers of a geometric series, dots-and-bars evaluation of coefficients.

2. Generating function for a sequence of numbers, closed formula for a recurrence relation generating function.

3. Operations on generating functions: substitution, multiplication by x, differentiation.

4. Extracting coefficients, partial fraction decomposition.

5. Cauchy product formula. Application: counting Catalan trees.

6. Interpretation of $A(x)B(x)C(x)\cdots$ where each $F(x)$ is an ordinary generating function satisfying $F(0) = 0$: number ways to separate $[n]$ into k consecutive intervals and do something to each interval.

6a. Example: organize books in alphabetical order on three distinct shelves, no shelf empty. Then $F(x)$ is the generating function for the sequence $1, 1, 1, \ldots$ and we obtain $\left(\frac{x}{1-x}\right)^3$. Another solution: compositions of n into 3 non-zero parts.

6b. Example: organize books in alphabetical order on three distinct shelves, no shelf empty, then choose a book on each shelf to put a bookmark in. Now $F(x) = \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$ and we obtain $\frac{x^3}{(1-x)^6}$.

7. Interpretation of $\frac{F(x)}{1-F(x)}$ where $F(x)$ is a ordinary generating function with $F(0) = 0$: number of ways to separate $[n]$ into $k \geq 1$ intervals and do something to each interval.

7a. Example: Total number of compositions of n: use any number of shelves. We obtain

$$\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x}.$$

Compare the earlier derivation.

7b. Example: organize books in alphabetical order on any number of distinct, no shelf empty, then choose a book on each shelf to put a bookmark in. Now $F(x) = \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$ and we obtain $\frac{x}{1-3x+x^2}$. An exact formula for the coefficients can be found using partial fraction decomposition.

8. Exponential generating function for a sequence.

9. Interpreting $A(x)B(x)C(x)\cdots Z(x)$ as an exponential generating function where each $F(x)$ is the exponential generating function for the sequence $a_1, a_2, a_3, \ldots$: form an ordered set partition of $[n]$ into k non-empty sets, then do something to each set as described by $F(x)$. First, form an ordered

set partition of given sizes: $\frac{n!}{e_1!e_2!\cdots e_k!}$ where $e_1 + e_2 + \cdots + e_k = n$. Next, do something to the first set in one of a_{e_1} ways, then something to the second set in one of b_{e_2} ways, etc. The result is

$$\frac{n!}{e_1!e_2!\cdots e_k!}a_{e_1}b_{e_2}\cdots z_{e_k}.$$

The exponential generating function for all of this is

$$\sum_{\substack{e_1+\cdots+e_k=n \\ e_i\geq 1}} \frac{a_{e_1}\cdots b_{e_k}}{e_1!e_2!\cdots e_k!}x^n = A(x)B(x)C(x)\cdots.$$

When unordered set partitions into k parts are called for, divide by $k!$.

9a. Example: find the exponential generating function for the number of ways to place n distinct balls into 3 identical boxes, no box empty: there is only one way to do something to the balls in the box, so the exponential generating function for balls in a box is $e^x - 1$. The exponential generating function for n balls into 3 identical boxes is therefore $\frac{(e^x-1)^3}{3!}$. Compare with the formula for $S(n,3)$ obtained using inclusion-exclusion.

9b. Example: find the exponential generating function for the number of ways to organize n distinct balls into 3 identical tubes, no tube empty: the number of tubes on $n \geq 1$ elements is $n!$, and its exponential generating function for tubes is $\sum_{n=1}^{\infty} \frac{n!}{n!}x^n = \frac{1}{1-x} - 1 = \frac{x}{1-x}$. So the exponential generating function for the n balls into 3 identical tubes is $\frac{1}{3!}\frac{x^3}{(1-x)^3}$. This enables us to count the number of configurations. A different counting argument would be to (1) choose a permutation (2) choose a spacing (3) divide by 3!.

9c. Example: find the exponential generating function for the number of permutations of n with 3 cycles: the number of cycles on $n \geq 1$ elements is $(n-1)!$, and its exponential generating function is $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Note that the derivative of this is $\frac{1}{1-x}$, so

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x).$$

So the exponential generating function for permutations with 3 cycles is $\frac{-(\ln(1-x))^3}{3!}$. The coefficients can be extracted using Mathematica.

9d. Example: same as 9c, but each cycle has at least 2 elements. The cycle generating function is now $-\ln(1-x)-x$.

10. Interpreting $e^{F(x)}$ as an exponential generating function where $F(x)$ is the exponential generating function for the sequence $a_1, a_2, a_3, \ldots$: no limit to the number of parts in the set partitions.

10a. Example: n distinct balls into any number of non-empty identical boxes. We obtain $e^{e^x-1}-1$, which yields a very complicated formula for $B(n)$.

10b. Example: n distinct balls into any number of identical tubes: $e^{\frac{x}{1-x}}-1$. Another complicated formula.

10c. Example: number of permutations: $e^{-\ln(1-x)}-1=\frac{x}{1-x}$ yields $n!$ permutations.

10d. Example: number of derangements: $e^{-\ln(1-x)-x}-1=\frac{e^{-x}}{1-x}-1$. Compare with the formula derived using inclusion-exclusion.

Exercises, p. 174: (1) $\sum_{k=0}^{\infty} a_{k+1}x^k = a_1 + a_2x + a_3x^2 + \cdots = \frac{F(x)}{x}$, $\sum_{k=0}^{\infty} a_k x^k = F(x)$, $\sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$. Solving, $F(x) = \frac{x}{(1-x)(1-2x)}$. Formula: $\frac{1}{(1-ax)(1-bx)} = \frac{1}{a-b}\left(\frac{a}{1-ax}-\frac{b}{1-bx}\right)$. Hence $F(x) = x\left(\frac{-1}{1-x}+\frac{2}{1-2x}\right)$ and $a_k = [x^{k-1}]F(x) = -1+2^k$. (2) $b_0 = a_0$ and $b_{n+1} = b_n + a_{n+1}$. Multiplying by x^n and summing over $n \geq 0$ we obtain $\frac{B(x)-b_0}{x} = B(x) + \frac{A(x)-a_0}{x}$. Hence $B(x) = \frac{A(x)}{1-x}$. A general formula for $\sum_{k=0}^{\infty} a_{k+i}x^k$ is $\frac{F(x)-(a_0+a_1x+\cdots+a_{i-1}x^{i-1})}{x^i}$. (3) skip (4) We are counting strings of $1s$ and $2s$ that sum to n. Recurrence relation: $f_n = f_{n-1}+f_{n-2}$ when $n \geq 3$ and $f_1 = 1$ and $f_2 = 2$. Equivalently $f_{n+3} = f_{n+2} + f_{n+1}$ when $n \geq 0$. We set $f_0 = 0$ for convenience. Hence $\frac{F(x)-0-1x-2x^2}{x^3} = \frac{F(x)-0-1x}{x^2} + \frac{F(x)-0}{x}$, $F(x) = \frac{x^2+x}{1-x-x^2} = -1 + \frac{1}{1-x-x^2}$. Formula: $\frac{1}{ax^2+bx+c} = \frac{1}{a(x-\alpha)(x-\beta)} = \frac{1}{a(\beta-\alpha)}\left(\frac{1/\alpha}{1-x/\alpha}-\frac{1/\beta}{1-x/\beta}\right)$. Note that $a\alpha\beta = c$, so once you find α and β you have $1/\alpha = (a/c)\beta$ and $1/\beta = (a/c)\alpha$. Simplifying, we obtain $\frac{1}{ax^2+bx+c} = \frac{1}{c(\alpha-\beta)}\left(\frac{\alpha}{1-(a/c)\alpha x}-\frac{\beta}{1-(a/c)\beta x}\right)$. (5) Let $F(x) = H(x)-1$ be the non-trivial generating function. This follows from $F(x)(1-3x+x^2) = 1$, since extracting $[x^{n+2}]$ we obtain $h_{n+2}-3h_{n+1}+h_n = 0$. (6) skip (7) Imagine a sequence of 10s, 5s, and 1s, adding up to n. The number of these is a_n. Organizing these by first number, this is equal to $a_{n-10}+a_{n-5}+a_{n-1}$, valid for $n \geq 11$. We can compute the lower numbers directly. This yields $a_{n+11} = a_{n+1}+a_{n+6}+a_{n+10}$ and is going to yield $\frac{p(x)}{1-x-x^6-x^{10}}$. (8) Geometric sequence has terms x^n. Differentiating and multiplying by x yields nx^n. Differentiating and multiplying by x yields n^2x^n. (9) The formula in the book is incorrect. A division of $[n]$ approach: circle the numbers in the subset. Then put a circled $n+1$ last and form the corresponding division of $[n+1]$. The first and last parts have size at least 1 and the other parts have size at least 3. So we are seeking the coefficient of x^{n+1} in $A(x)C(x)+A(x)B(x)C(x)+A(x)B(x)^2C(x)+\cdots = A(x)\frac{1}{1-B(x)}C(x)$ where $A(x)=\frac{x}{1-x}$, $B(x)=\frac{x^3}{1-x}$, $C(x)=\frac{x}{1-x}$. In other words, the coefficient of x^n in $\frac{1}{x}A(x)\frac{1}{1-B(x)}C(x)=\frac{x}{(1-x)(1-x-x^3)}$. Mathematica yields $x+2x^2+3x^3+5x^4+8x^5+12x^6+18x^7+27x^8+40x^9+59x^{10}+\cdots$. (10) Count partitions with largest part k instead using a segmenting approach. To ensure at least one partition component of each size, add the parts $1,2,\ldots,k-1$. Now use the generating functions $\frac{x}{1-x}$, $\frac{x^2}{1-x^2}$, ... $\frac{x^k}{1-x^k}$. Multiplying, we are segmenting $[n]$ into subintervals of sizes in $\{1,2,3,\ldots\}$, $\{2,4,6,\ldots\}$, ..., $\{k,2k,3k,\ldots\}$. We want the coefficient of $x^{n+1+2+\cdots+(k-1)}$. Equivalently, we want the coefficient of x^n in $\frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$. (11)–(18) skip (19) We are counting permutations with cycles of length ≤ 2. We need an ordered set partition of $[n]$ with an arbitrary number of non-empty parts of size ≤ 2. This yields $e^{\frac{x}{1!}+\frac{x^2}{2!}}$. (22) Segment $[n]$ into parts of sizes $\{1,3,5,\ldots\}$. This yields $\frac{A(x)}{1-A(x)}$ where $A(x)=\frac{x}{1-x^2}$. Simplifying, we obtain $\frac{x}{1-x-x^2}$. This yields $b(n)=f(n-1)$ for $n\ge 2$.

Supplementary Exercises, p. 176: (23) The recurrence relation implies $\frac{f(x)-1}{x}=3f(x)+\frac{1}{1-2x}$. Hence $f(x)=\frac{1-x}{1-5x+6x^2}=\frac{-1}{1-2x}+\frac{2}{1-3x}$. Hence $a_n=-1(2^n)+2(3^n)$. (24) $\frac{f(x)-1-4x}{x^2}=8\frac{f(x)-1}{x}-16f(x)$, $f(x)=\frac{1}{1-4x}$, $a_n=4^n$. (25) The recurrence relation is $a_0=50$ and $a_{n+1}=2a_n+1000$ for $n\ge 0$. This yields $\frac{f(x)-50}{x}=2f(x)+\frac{1000}{1-x}$, $f(x)=\frac{50+950x}{(1-x)(1-2x)}=\frac{-1000}{1-x}+\frac{1050}{1-2x}$, $a_n=-1000+1050(2^n)$. (26) skip (27) $\frac{a_{n+1}}{(n+1)!}=\frac{a_n}{n!}+2$ with $a_0=0$. Setting $b_n=\frac{a_n}{n!}$ we obtain $b_{n+1}=b_n+2$ with $b_0=0$. $\frac{B(x)-0}{x}=B(x)+\frac{2}{1-x}$, $B(x)=\frac{2x}{(1-x)^2}$, $b_n=2n$, $a_n=(2n)n!$. (28) $\frac{a_n}{n!}=\frac{a_{n-1}}{(n-1)!}+\frac{a_{n-2}}{(n-2)!}$ yields $b_n=b_{n-1}+b_{n-2}$ or $b_{n+2}=b_{n+1}+b_n$ with $b_0=b_1=1$. $\frac{B(x)-1-x}{x^2}=\frac{B(x)-1}{x}+B(x)$, $B(x)=\frac{1}{1-x-x^2}$. (29) $\frac{a_{n+1}}{(n+1)!}=\frac{a_n}{n!}+\frac{1}{n+1}$, $b_{n+1}=b_n+\frac{1}{n+1}$, $\frac{B(x)-0}{x}=B(x)+\sum_{n=0}^{\infty}\frac{x^n}{n+1}$, $B(x)=\frac{\sum_{n=1}^{\infty}\frac{x^n}{n}}{1-x}$, $b_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$, $a_n=n!(1+\frac{1}{2}+\cdots+\frac{1}{n})$. (30)–(32) skip (33) Choose an unordered set partition where each part has size k. The exponential generating function for the parts of size ≥ 1 is $\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots=e^x-1$. So we obtain $\frac{(e^x-1)^k}{k!}$. (34) We are segmenting $[n]$ into an arbitrary number subintervals of length 2 or 3. So we obtain $1+\frac{x^2+x^3}{1-(x^2+x^3)}=\frac{1}{1-x^2-x^3}$. (35)

We are segmenting $[n]$ into subintervals of size 1 or 2. There are two things we can do to a subinterval of size 1: color it red or blue. there are three things we can do to a segment of size 2: color it green or yellow or white. So $H(x) = \frac{2x+3x^2}{1-2x-3x^3}$. (36) The typical sequence of length ≥ 3 begins with B or AB. This yields $h_n = h_{n-1} + h_{n-2}$ for $n \geq 3$ with $h_1 = 2$ (the sequences A and B) and $h_2 = 3$ (the sequences AB, BA, BB). In other words, $h_{n+3} = h_{n+2} + h_{n+1}$. This yields $\frac{H(x)-0-2x-3x^2}{x^3} = \frac{H(x)-0-2x}{x^2} + \frac{H(x)-0}{x}$, $H(x) = \frac{2x+x^2}{1-x-x^2}$. (37) –(38) skip (39) Choose an ordered set partition of $[n]$ into 2 sets, and with each set form a permutation of its elements. The exponential generating function for permutations is $\frac{1!}{1!}x + \frac{2!}{2!}x^2 + \cdots = \frac{x}{1-x}$. So $T(x) = \left(\frac{x}{1-x}\right)^2$. This implies $\frac{t_n}{n!} = n-1$ and $t_n = n!(n-1)$. Another proof: form a permutation of $[n]$, then decide how to separate into two non-empty intervals. There are $n!$ was to form a permutation, and $n-1$ ways to separate (separation occurs after book 1 or book 2 or ... or book $n-1$). (40) Derangements are permutations with cycles of length ≥ 2. First form an unordered set partition of arbitrary size, then arrange the elements of each set into a cycle. The exponential generating function for arranging elements into a cycle is $\frac{1!}{2!}x^2 + \frac{2!}{3!}x^2 + \cdots = \sum_{k=2}^{\infty} \frac{x^k}{k} = \log(\frac{1}{1-x}) - x$, so $D(x) = e^{(\log(\frac{1}{1-x})-x)} = \frac{e^{-x}}{1-x}$. (41) It suffices to show that the exponential generating function for the implied recurrence relation is $\frac{e^x}{1-x}$. (42) skip (43) Form an ordered set partition into three sets (empty sets allowed). The first set must have odd size, the second set must have even size, and the third set can have any size. We must form a permutation of each set since the people are arranged in a line. The exponential generating function for this process is $A(x)B(x)C(x)$, where $A(x)$ is the exponential generating function for permutations of sets of even size, $B(x)$ is the exponential generating function for permutations sets of odd size, and $B(x)$ is the exponential generating function for permutations sets of arbitrary size. We have $A(x) = 1 + \frac{2!x^2}{2!} + \frac{4!x^4}{4!} + \cdots = \frac{1}{1-x^2}$, $B(x) = \frac{1!x^1}{1!} + \frac{3!x^3}{3!} + \cdots = \frac{x}{1-x^2}$, $C(x) = \frac{0!x^0}{0!} + \frac{1!x^1}{1!} + \cdots = \frac{1}{1-x}$. Given $F(x) = A(x)B(x)C(x) = \frac{x^3}{(1-x^2)(1-x)} = \frac{x}{(1-x)^2(1+x)} = x\left(\frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x}\right)$, we have $\frac{f_n}{n!} = [x^{n-1}]\left(\frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x}\right) = (1/4)(1) + (1/2)\binom{n-1+1}{1} + (1/4)(-1)^{n-1} = \frac{1+2n-(-1)^n}{4}$. Hence $f_n = n!\frac{1+2n-(-1)^n}{4}$. (44) Form an ordered set partition into two sets, the first non-empty, the second possibly empty. If the first set has even size, do nothing, but if it has odd size, choose an even-sized subset. The exponential generating function for this process is $F(x) = A(x)B(x)$

where $A(x) = \frac{2^0}{1!}x^1 + \frac{2^2}{3!}x^3 + \frac{2^4}{5!}x^5 + \cdots = \frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$ and $B(x) = e^x$. Hence $F(x) = \frac{e^{3x}-e^{-x}}{4}$, $f_n = \frac{3^n-(-1)^n}{4}$. (45) – (50) skip

Chapters 9 and 10: Graph Theory and Trees (15 meetings)

Graph Theory Concepts: Graph, vertices, edges, vertex degree, trail, walk, closed trail, Eulerian Trail, path, connected graph, complete graph, connected components, Hamilton Cycle, Hamilton Path, directed graph, directed trails/walks/paths, strongly-connected directed graph, in-degrees and out-degrees, balanced directed graph, tournament, graph isomorphism.

1. Definitions.

Graph: (V, E).

Vertex degree: number of edges containing vertex.

Trail of length k: $(e_1, e_2, \ldots, e_k)$, distinct edges, consecutive edges sharing a vertex.

Walk of length k: $(e_1, e_2, \ldots, e_k)$, edges, consecutive edges sharing a vertex, edges not necessarily distinct.

Closed trail of length k: return to vertex in e_1 that is not in e_2. Also called a cycle.

Eulerian Trail: every edge in graph appears.

Path of length k: trail, no self-intersections.

Trail/walk/path between vertices of length k: $(x_1, e_1, x_2, e_2, \ldots, x_k, e_k, x_{k+1})$ $x_i \in e_i$, $1 \leq i \leq k$, and $x_{k+1} \in e_k$. This allows us to talk about trails/walks/paths of length 0 and between a vertex and itself.

Connected graph: every pair of vertices are the endpoints of some walk. A single-vertex graph is considered to be connected.

Connected component of a graph: form an equivalence relation on V: $x \equiv y$ iff x, y are the endpoints of a walk in the graph. Now form the equivalence classes. A connected component of a graph $G = (V, E)$ is $C_i = (V_i, E_i)$ where V_i is an equivalence class and $E_i = \{e \in E : e \subseteq V_i\}$.

Complete graph: $K_n = ([n], \binom{[n]}{2})$.

Hamilton Cycle: A closed trail (cycle) in the graph that contains every vertex in the graph.

Hamilton Path: A path in the graph that contains every vertex in the graph.

Directed graph: a graph with edge directions.

Directed trail/walk/path from x to y: edge joining x_i to x_{i+1} points from x_i to x_{i+1}.

Strongly-connected graph: For every pair of vertices x and y there is a directed walk from x to y and a directed walk from y to x.

Out-degree of a vertex x: number of edges of the form $x \to y$.

In-degree of a vertex x: number of edges of the form $y \to x$.

Balanced directed graph: $indegree(x) = outdegree(x)$ for every vertex x.

Tournament: K_n with edge directions.

Graph isomorphism: two graphs G and H are isomorphic iff they can be superimposed.

2. Theorems.

1. **Theorem:** : Let G be a graph. Then $\sum_{x \in V} deg(x) \equiv 0 \bmod 2$.
Proof: Place a dot on either end of every edge. $2e$ dots. Dots per vertex x: $deg(x)$.

2. Theorem: : Let G be a graph. Let V_0 be the set of vertices with degree $\equiv 0 \bmod 2$. Let V_1 be the set of vertices with degree $\equiv 1 \bmod 2$. Then $|V_1| \equiv 0 \bmod 2$.

Proof: $0 \equiv \sum_{x \in V_0} deg(x) + \sum_{x \in V_1} deg(x) \equiv \sum_{x \in V_1} 1 = |V_1| \bmod 2$.

3. **Theorem:** : If $V = V_0$ and $|E| > 0$ then G contains a cycle.

Proof: Extend an edge to a path of maximal length. The endpoint has degree ≥ 2, so another edge, which by maximality can only point to another vertex on the graph. This yields a cycle.

4. **Theorem:** : Every connected component of G has a closed Euler Trail iff $V = V_0$.

Proof: Assume every connected component of G has closed Euler Trail. Walking along Euler Trails, distribute dots as follows: drop a dot just after departing a vertex and just before arriving at a vertex. Except for the first vertex in a trail, every vertex gets 2 dots at a time, and since every edge is used, every vertex gets one dot per edge it belongs to. Hence even degree.

The first vertex gets 1 dot on departure, then dots in pairs, then 1 more dot on the last arrival, so an even number of dots. It has even degree.

Conversely, suppose $V = V_0$ in G. Proceed by induction on number of edges in G. Base case: $e = 0$. Every component has a single vertex, hence a closed Euler trail. $e = 1$: not possible. $e = 2$: not possible. $e = 3$: graph must be a triangle plus any number of isolated vertices. Induction hypothesis: For graphs with up to e vertices, the statement is true. Now let G be a graph with $e + 1$ vertices satisfying $V = V_0$. By Theorem 3, G has a cycle C. Let $H = G - C$. Then H satisfies $V = V_0$, and has $e - 2$ edges, so every connected component of H has a closed Euler Trail. C glues some of these together into a closed trail, so every edge in G belongs to exactly one closed trail.

5. **Theorem:** : Let G be a connected graph. Let $s \neq t$ be vertices in G. Then there is an Euler Trail with endpoints s and t iff $V_1 = \{s, t\}$.

Proof: Assume the Euler Trail exists. Create a new graph H by adding to G a new vertex x and the edges $\{x, s\}$ and $\{x, t\}$. H has a closed Euler Trail (walk through all edges from s to t, then travel to x, then travel to s). $V = V_0$ in H implies $V_1 = \{s, t\}$ in G.

Conversely, assume $V_1 = \{s, t\}$. Create H as before. It satisfies $V = V_0$ and is connected, so has an Euler Trail. Picking up the trail at x it travels first to s (WLOG) and cannot return to x until it has visited every other edge first (otherwise we get stuck at x before visiting all edges). The last edge is t to x. This implies an Euler Trail in G from s to t.

6. **Theorem:** : Let G be a graph with n vertices. If $deg(x) \geq \frac{n-1}{2}$ for each vertex x then G is connected.

Proof: Suppose G has at least two connected components. Choose $x_1 \in C_1$ and $x_2 \in C_2$. C_1 has at least $1+deg(x_1)$ vertices and C_2 has at least $1+deg(x_2)$ vertices, therefore $C_1 \cup C_2$ has at least $2+deg(x_1)+deg(x_2) \geq 2+\frac{n-1}{2}+\frac{n-1}{2} = n + 1$ vertices. Contradiction. Therefore G must be connected.

7. **Theorem:** : Let G be a graph with $n \geq 3$ vertices. If $deg(x) \geq \frac{n}{2}$ for each vertex x then G has a Hamilton Cycle.

Proof: Pick a path of maximum possible length k in the graph and say that it has vertices $x_0, x_1, \ldots, x_k$. By maximality of the path length, every edge out of x_0 has endpoint in $\{x_1, x_2, \ldots, x_k\}$. Similarly, every edge out of x_k has an endpoint in $\{x_0, x_1, \ldots, x_{k-1}\}$. Now suppose that whenever $x_k - x_i$

is an edge, $x_0 - x_{i+1}$ is not an edge. This implies that if $deg(x_k) = a$ then $deg(x_0) \leq k - a$. Therefore

$$k = a + (k - a) = deg(x_k) + (k - a) \geq deg(x_k) + deg(x_0) \geq n.$$

However, this implies there are $\geq n + 1$ vertices on the path, which is too many. So there must be an instance where $x_k - x_i$ and $x_0 - x_{i+1}$ are both edges in the graph. Use this to construct a cycle of $k + 1$ vertices: x_0 to x_i along the path, then to x_k, then backward along the path to x_{i+1}, then back to x_0. Our last step is to show that $k + 1 = n$. If $k + 1 < n$ then there has to be an edge from a cycle vertex to a non-cycle vertex, otherwise no path in the graph can escape the cycle and travel to a vertex off the cycle. This edge, plus the cycle edges, implies the existence of a path of length $k + 1$ in the graph: contradiction. Hence $k + 1 = n$ and the cycle we constructed incorporates every vertex.

8. **Theorem:** : A directed graph G decomposes into closed Eulerian Trails iff it is balanced and and each underlying connected component is strongly connected.

Proof: If the decomposition exists then dot-depositing implies balanced and the underlying connected components must be strongly connected because we can circulate around the trails to form directed paths. Conversely, if the graph is balanced and the underlying connected components are strongly connected, use a directed version of the proof (requires a directed version of Theorem 3). Note that for the converse we don't need the assumption that the underlying components are strongly connected (so this condition is forced by the balanced hypothesis).

9. **Theorem:** : Every tournament has a Hamilton path.

Proof: We will prove that there exist directed paths of length k edges for every $k \leq n-1$ by induction on k. Base case: pick an arbitrary directed edge. Induction hypothesis: for some k there exists a directed path $x_0 \to x_1 \to \cdots \to x_k$. If $k = n - 1$ we're done, so suppose $k < n - 1$. We will show that there is a directed path of length $k+1$. Pick an arbitrary vertex y not on the path and form the sequence $D_0, D_1, \ldots, D_k$, where each $D_i \in \{T, F\}$, where $D_i = T$ means $x_i \to y$ and $D_i = F$ means $y \to x_i$. If our sequence contains a TF in it then our new directed path is $x_0 \to\to x_i \to y \to x_{i+1} \to\to x_k$. If our sequence does not contain a TF in it, it is either T^{k+1} or $F^a T^b$ for $a \geq 1$, so it either ends with T or begins with F. If it it begins with T, append $x_k \to y$. It it begins with F, prepend $y \to x_0$.

10. **Theorem:** : A tournament has a Hamilton cycle iff it is strongly connected.

Proof: If a tournament has a Hamilton cycle then it is strongly connected: just follow the cycle from one vertex to another one. Conversely, suppose a tournament is strongly connected. The tournament has at least one cycle in it: find a path of maximal length. The last vertex on the path has an edge out of it because there has to be a path back to the first vertex. This edge points to a vertex along the maximal path, so there is a cycle. Now let C be a cycle with the maximum possible number of vertices in it. We will argue that it must contain all the vertices of the tournament. Suppose it doesn't. Then there must be an edge pointing from a cycle vertex c to a non-cycle vertex x, otherwise there is no way to leave the cycle and travel to a non-cycle vertex. Now if the cycle is $c \to c' \to c'' \to \cdots$, then $c' \to x$, otherwise we could create a longer cycle. Similarly, $c'' \to x$, $c''' \to x$, etc. All the cycle vertices point to x. Let X be the set of all vertices of this description, namely non-cycle vertices x such that $c \to x$ for all $c \in C$. If every edge out of every $x \in X$ pointed back into X, there would be no way to escape X, so no way to return to C, which contradicts strong connectedness. So there has to be an edge of the form $x \to y$ where $x \in X$ and $y \notin X$. We know $c \to x$ because every cycle vertex points to x. We can't have $y \to c'$ by maximality of the cycle, and we can't have $c' \to y$ since once any one cycle vertex points to y, they all do. So there can't be any edge between y and c', which contradicts the fact that our graph is a tournament. So C must include all vertices.

Chapter 9 Exercises, p. 202: (1) Pick an arbitrary vertex in each connected component and label them with consecutive names: $x_1, x_2, \ldots$. Repeat the following step until all vertices have been labeled: find the smallest labeled vertex that has edges to unlabeled vertices and direct all of these edges outward, labeling the vertices on the other end of these edges with the smallest available unused label. When done, all edges point from a smaller vertex to a larger vertex, so there can't be any directed cycles. (2) No: C_3. (3) Skip (4) Let Δ be the maximum vertex degree. Adding the degrees together, $27 \le 9\Delta$, therefore $3 \le \Delta$. If $\Delta = 3$ then all 9 vertex degrees are 3. Impossible: there must be an even number of odd degrees. So $\Delta \ge 4$. (5) For each subset S of 4 vertices let $E(S)$ denote the set of edges among vertices in S. Every $|E(S)|$ is between 0 and $\binom{4}{2} = 6$, and we want one of them to equal 6. Let the maximum value of $|E(S)|$ be Δ. Add all these numbers up. What do we get? Label an edge with $S, S', S'' \cdots$ whenever it belongs to $S, S', S'', \ldots$. Then

$$\sum_{\substack{S \subseteq V \\ |S|=4}} |E(S)|$$

is the total number of edge labels. Each edge is labeled with $\binom{8}{2}$ different labels because that's how many 4-element subsets its vertices belongs to, given $|V| = 10$. Hence

$$\sum_{\substack{S \subseteq V \\ |S|=4}} |E(S)| = 38\binom{8}{2} = 1064.$$

On the other hand,

$$\sum_{\substack{S \subseteq V \\ |S|=4}} |E(S)| \leq \binom{10}{4}\Delta,$$

so we get $1064 \leq 210\Delta$, $\Delta \geq 5.06667$, which implies $\Delta = 6$ since Δ is an integer. So there is an S with $|E(S)| = 6$. (6) The graph decomposes into closed Eulerian Trails. Walks around the edges and assign alternating color edges. (7) We must decide which of the edges in K_n belong to the graph. 2 choices for each edge. So $2^{\binom{n}{2}}$. (8) Think of a graph isomorphism between G to H as a function $f : V(G) \to V(H)$. After the graphs are superimposed, $f(x) =$ vertex in V_H that x is on top of. An automorphism of a graph with vertex set $[n]$ can be thought of as a permutation that preserves structure. Counting automorphisms: (a) Any vertex permutation of K_n yields K_n, so $n!$ (b) All we can do is rotate, or rotate and then flip, so $2n$. (c) All we can do is reverse order. So 2 if $n \geq 2$. (d) In S_n the central vertex cannot change but the outer vertices can be permuted at will, so $(n-1)!$. (9) Say that two graphs with vertex set $[8]$ are equivalent iff they are isomorphic. This is an equivalence relation. Organize into equivalence classes $C_1, C_2, \ldots, C_h$. We want $h \geq 6600$. We have $|C_1| + |C_2| + \cdots |C_h| = 2^{\binom{8}{2}}$. To show that h is large we must show that each $|C_i|$ is small. Given any graph in C_i, we can obtain the others in C_i by applying an isomorphism to these, and there are $\leq 8!$ of them. So we have $2^{\binom{8}{2}} = |C_1| + \cdots + |C_h| \leq h(8!)$, therefore $h \geq 2^{\binom{8}{2}}/8! = 6657.63$. (10) Yes, by Theorem 2 above. (11) (a)

No: butterfly graph. (b) No: K_4. (12) Vertex degree sum yields $44 = |V|deg$, so $|V| \in \{1, 2, 4, 11, 22, 44\}$. Not all number are possible, because $\binom{|V|}{2} \geq 22$. We actually need $|V| \in \{11, 22, 44\}$. But we're not done until we actually construct the graph. If $|V| = 11$ then all degrees are 4. Use Harary construction. If $|V| = 22$ then all degrees are 2. Big cycle. If $|V| = 44$ then all degrees are 1, which implies 22 disjoint edges, which is not connected. (13) skip (14) If it does, then exploring the possibilities we get a contradiction in every case. So it doesn't. (15) skip (16) The number of parts of the partition is the number of non-isolated neighbors, say k. The largest part of the partition is the largest degree, which is at most $k - 1$. So no. (17) skip (18) skip (19) skip (20) skip (21) $C_3 + C_3$ and C_6 both have degree sequence 2^6. (22) A graph is a collection of connected components. To create one, form a set partition on $[n]$ and construct a connected graph on each. Hence is $F(x)$ is the exponential generating function for all graphs, $F(x) = e^{C(x)}$. The coefficients of $F(x)$ are $\frac{2^{\binom{n}{2}}}{n!}$.

Supplementary Exercises, p. 205: (23) (a) Number of directed graph with k edges: First choose k edges out of $N = \binom{n}{2}$ available, then assign an orientation to these. This yields $\binom{N}{k}2^k$. Total is $\sum_{k=0}^{N} \binom{N}{k}2^k = 3^N = 3^{\binom{n}{2}}$. (b) This is equal to the last term in the sum, namely $2^N = 2^{\binom{n}{2}}$. (24) skip (25) skip (26) Yes. Form directed path of maximal length. To avoid cycle, last vertex must have out-degree 0. (27) By induction on $n \geq 1$, $n = 2$: There are two non-isomorphic graphs on $[2]$, and $2 = \binom{2}{2} + 1$. Now assume one can find $\binom{n}{2} + 1$ non-isomorphic graphs on $[n]$. We can consider these as non-isomorphic graphs on $[n+1]$ by adding $n+1$ as an isolated vertex. We need to find at least n more. So we can try to find n non-isomorphic graphs on $[n+1]$ that have no isolated vertex. Try $P_{n+1}, P_{n+1} + 12, P_{n+1} + 12 + 13, \ldots, P_{n+1} + 12 + 13 + \cdots + 1(n+1)$. (28) A path is a walk with no self-intersections. Given a walk from A to B, the first time you encounter a vertex a second time, cut out all edges of the walk after the first encounter and before the second encounter. Obtain a shorter AB walk. Repeat as necessary until no self-intersections. (30) In an n vertex graph, degrees range from 0 to $n - 1$. If all degrees are different there are one of each, including $n - 1$, which precludes 0. Contradiction. (31) We have a graph in which $deg(x) = deg(y)$ and $x \neq y$ implies $N(x) \cap N(y) = \emptyset$. We wish to prove that some vertex has degree 1. Use induction on number of edges. If there is just 1 edge, no problem. Now assume the property is true for n edges. Consider

$n+1$ edges. Delete an arbitrary edge. If we generate a graph satisfying the same condition, then the induction hypothesis guarantees the existence of a degree 1 vertex in the smaller graph. If it is not an endpoint of the edge we deleted, then the original graph has a degree 1 vertex. But if it is an endpoint of the edge we deleted, then the vertex it is joined to has degree 1 in the original graph. Now suppose we delete an edge and do violate the condition. Then there is an edge in the original graph with endpoints of degree k and $k-1$ for some $k \geq 2$. Now if we choose our edge carefully we can avoid this second scenario: just choose an edge containing a vertex of minimal degree. (31) skip (32) skip (33) Model what has happened so far as a graph on 10 vertices with 11 edges. We want a vertex to have degree ≥ 3. If all degrees are ≤ 2 then the vertex degree sum is at most 20, which implies at most 10 edges. Contradiction. (34) Just do it. (35) In other words, find the smallest value of $n \geq 2$ such that there exists a graph with only the trivial automorphism. This is a trial and error process. Probably $n = 4$. (36) First consider one r-cycle C. Draw it so that vertex 1 is on top and vertex 2 is clockwise from 1. Every automorphism of C is determined by the resulting positions of 1 and 2. There are r choices for where to send 1 and 2 choices for where to send 2, given that it is joined to 1. The remaining vertex positions are determined. So there are $2r$ automorphisms of C. Now consider cycles $C_1, \ldots, C_k$. Any permutation of vertices that preserves the graph will have to do something to each cycle and permute the order of the cycles. Hence there are $k!(2r)^k$ automorphisms. (37) skip (38) skip (39) Number of directed cycles starting at 1: $(n-1)!$. But as distinct graphs, only $(n-1)!/2$. (40) Any cycle must have an equal number of vertices from A and from B, so $m = n$ is required for existence. Counting directed cycles starting at a_1, we will encounter the vertices in A in $(n-1)!$ ways and the vertices in B in $n!$ ways, for a total of $(n-1)!n!$. Number of distinct subgraphs is $(n-1)!n!/2$. (41) $Q_2 = C_4$, so it has a Hamilton cycle. Assuming Q_n does, Q_{n+1} can be described as two copies of Q_n plus edges joining $0v$ to $1v$ where $v \in Q_n$. Find a Hamilton path in each copy of Q_n with endpoints u_1, u_2 on one side and endpoints v_1, v_2 on the other side, then create a Hamilton cycle in Q_{n+1} by introducing edges $0u_1 - 1v_1$ and $1v_2 - 0u_2$. (42) Q_2 has 2, which is more than enough. Assuming Q_n has at least $n!/2$, the construction in (41) yields at least $n(n!/2)^2$ Hamilton cycles in Q_{n+1}, which is also enough (first few cases must be checked). (43) skip (44) skip (45) Find path of maximal length. Last vertex has k edges to vertices on the path. So one of them reaches back at least k vertices, yielding a $(k+1)$-cycle. (46) Suitably modify Theorems

6 and 7 above. (47) To create a counter-example, add a vertex to K_{n-1} and create a single edge out of this vertex. (48) skip (49) skip (50) skip.

Tree concepts: Tree, forest, number of edges in a tree or forest, number of trees, spanning trees, minimal weight spanning trees, Kruskal's algorithm, adjacency matrix, counting walks using the adjacency matrix, counting spanning trees using the Matrix-Tree Theorem.

1. Definitions.

Tree: Connected graph, no cycles.

Forest: Graph, no cycles. Each component is a tree.

Leaf: Degree one vertex.

Bridge edge: An edge whose removal from a graph increases the number of connected components.

Rooted tree: tree with one vertex designated the root.

Doubly-rooted tree: tree with a circled vertex and a boxed vertex (they can be the same vertex).

Spanning tree: subgraph which contains all vertices (spanning) and is a tree.

Adjacency matrix: 0-1 matrix with rows and columns representing vertices and entries representing edges.

Weighted graph: edges have positive weights. Alternatively, complete graph with non-negative weights.

Kruskal's Algorithm: Let G be a connected graph of order n with edge weights. Choose an edge e_1 of smallest possible weight. Having found the acyclic collection of edges $e_1, \dots, e_k$, choose e_{k+1} of minimum weight that extends the acyclic collection. Keep on going until no longer possible.

2. Theorems.

1. **Theorem:** Let G be connected. e is a bridge edge iff e does not belong to a cycle.

Proof: If e belongs to a cycle then its removal does not disconnect G so it is not a bridge. Conversely, if e is not a bridge then $G-e$ is connected, so there is a path between the endpoints of e in $G-e$, forming a cycle containing e.

2. **Theorem:** A graph is a tree iff every pair of vertices is joined by a unique path.

Proof: Tree implies no cycles implies unique paths. Unique paths implies no cycles implies tree.

3. **Theorem:** Every nontrivial tree contains at least two leaves.

Proof: Find a maximal path. Endpoints are leaves, otherwise there's a cycle.

4. **Theorem:** Let G be a connected graph. Then G is a tree if and only if $e = v - 1$.

Proof: Let G be a tree. We will prove $e = v - 1$ by induction on v. If $v = 1$ then $e = 0$ and we are done. Assume when $v = n$, $e = n - 1$. Now consider $v = n + 1$. Find a leaf vertex. The edge it belongs to cannot be a bridge, so its deletion leaves a connected graph which is still acyclic, hence still a tree. This tree has n vertices, hence $n - 1$ edges, therefore the original graph has n edges.

Conversely, assume only that G is connected and satisfies $e = v - 1$. Since deletion of cycle edges leaves a connected graph and all vertices intact, we can successively delete cycle edges until what's left has none and is still connected. At this point we have arrived at a tree, which has $v - 1$ edges. In other words, we didn't delete any edges to begin with, so G itself is a tree.

5. **Theorem:** A forest has $v - e$ trees in it.

Proof: Say that the forest has k trees in it. Adding $k - 1$ edges produces a tree with $E = e + k - 1$. The identity $E = v - 1$ implies $e + k - 1 = v - 1$ implies $k = v - e$.

6. **Theorem:** Let G be acyclic. Then $e = v - 1$ implies G is a tree.

Proof: G is a forest consisting of $v - e = 1$ trees.

7. **Theorem:** Any connected collection of n edges encompassing $n + 1$ vertices is a tree.

Proof: Let G be the resulting graph. By Theorem 4 it is a tree.

8. **Theorem:** Let G be a graph and assume that every vertex degree is $\geq k$. Then G contains an isomorphic copy of every tree on k edges.

Proof: By induction on k. If $k = 0$ then G consists of isolated vertices, each of which is a tree on 0 edges. If $k = 1$ then G consists of isolated edges, each of which is a tree on 1 edge. Now consider $k \geq 2$. Let T be an arbitrary tree with k edges. Delete a leaf edge $\{t, t'\}$ from T, producing T' with $k-1$ edges. By the induction hypothesis, G contains a copy of T'. At the copy of t' there must be an edge of G that does not appear in the copy of T'. Use another edge of G to produce a copy of the edge $\{t, t'\}$. So there is a copy of T in G.

9. **Theorem:** There are n^n doubly-rooted trees in K_n, hence n^{n-2} trees in K_n.

Proof: Given a doubly-rooted tree, form a forest of rooted trees by deleting the edges along the path between the circled and boxed vertex and using the vertices along this path as the roots of the trees in the forest. Orient all the remaining edges towards the roots and interpret as a function from the non-root vertices to the non-leaf vertices. Extend the inputs of the function to the root vertices by defining the permutation in two-line format where the top line is the list of root vertices in increasing order and the bottom line is the list of root vertices in the order in which they appear along the path. We have created a function $f : [n] \to [n]$. Different doubly-rooted trees give rise to different functions. The mapping between doubly-rooted trees and functions is surjective: given a function $f : [n] \to [n]$, create a directed graph D with vertex set $[n]$ and edges $i \to f(i)$. The function f combined with the cycle vertices C form a permutation of σ of C which can be represented in two-line format. The order of the vertices in the bottom line of σ can be represented by a path graph, and we can create a doubly-rooted tree which gives rise to f by adjoining to this path the edges in D out of non-C vertices.

10. **Cauchy-Binet Theorem:** Assume $p \leq q$. Let $A = (a_{ij})$ be an $p \times q$ matrix, let $B = (b_{ij})$ be a $q \times p$ matrix, and write $AB = C = (c_{ij})$. Then

$$\det(AB) = \det(C_1, \ldots, C_p) = \det(\sum_{i=1}^{q} b_{i1} A_i, \ldots, \sum_{i=1}^{q} b_{ip} A_i) =$$

$$\sum_{1 \leq i_1, \ldots, i_p \leq q} b_{i_1 1} \cdots b_{i_p p} \det(A_{i_1}, \ldots, A_{i_p}) =$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq q} \sum_{\sigma \in \mathcal{S}_p} b_{i_{\sigma(1)} 1} \cdots b_{i_{\sigma(p)} p} \det(A_{i_{\sigma(1)}}, \ldots, A_{i_{\sigma(p)}}) =$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq q} \sum_{\sigma \in \mathcal{S}_p} b_{i_{\sigma(1)} 1} \cdots b_{i_{\sigma(p)} p} \operatorname{sgn}(\sigma) \det(A_{i_1}, \ldots, A_{i_p}) =$$

$$\sum_{I \in \binom{[q]}{p}} \det(A_I)\det(B_I)$$

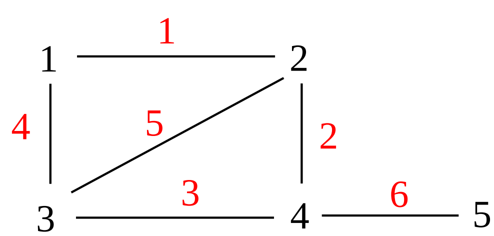
where for a subset I of $[q]$ of size p, A_I is the submatrix of A using the p columns from I and B_I is the submatrix of B using the p rows from I.

11. **Matrix-Tree Theorem:** Given a graph G with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$, the number of spanning trees in G is $\det(CC^T)$, where

$$C = ((-1)^{\chi(x_i = \min e_j)}\chi(x_i \in e_j))$$

and $1 \leq i \leq n-1$ and $1 \leq j \leq m$.

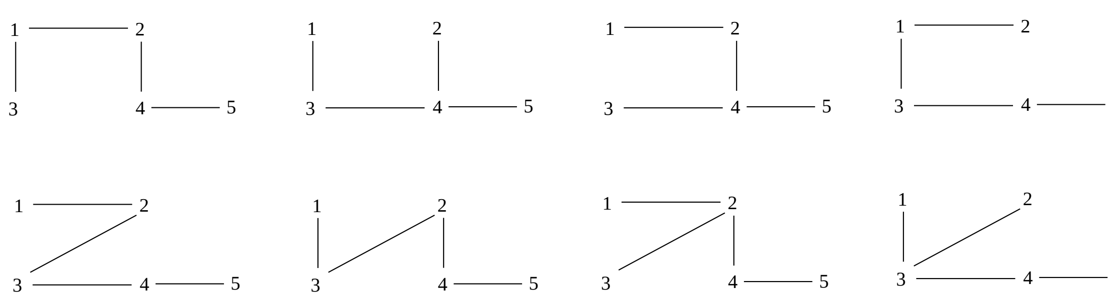
Example:

$G =$ 

$$C = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\det(CC^T) = 8$$

Spanning trees:



Proof of the Matrix-Tree Theorem: The number of spanning trees of G is

$$\sum_{H\in\binom{E}{n-1}} \chi(H \text{ is a spanning tree}).$$

We have

$$\det(CC^T) = \sum_{I\in\binom{[m]}{n-1}} \det(C_I)\det(C_I^T) = \sum_{I\in\binom{[m]}{n-1}} \det(C_I)^2.$$

We will prove that

$$\det(C_I)^2 = \chi(\{e_i : i \in I\} \text{ is a spanning tree})$$

for each $I \in \binom{[m]}{n-1}$.

Let $I \in \binom{[m]}{n-1}$ be given. Name the corresponding edges $f_1, \ldots, f_{n-1}$. Then the ij-entry of C_I is 0 if $x_i \notin f_j$ and is ± 1 if $x_i \in f_j$. These edges form a spanning tree if and only if they are connected and encompass n vertices.

Case 1: $\{f_1, \ldots, f_{n-1}\}$ does not incorporate all n vertices. If x_n is isolated then each column of C_I has a 1 and a -1 in it, so the sum of its rows is the 0 vector, so its rows are linearly dependent and $\det(C_I) = 0$. If some other vertex x_k is isolated then row k in C_I is the 0 vector, which again implies $\det(C_I) = 0$.

Case 2: $\{f_1, \ldots, f_{n-1}\}$ encompasses all n vertices but is not connected. Each component has at least two vertices. The sum of all the rows corresponding to vertices in a component not containing x_n is 0, hence the rows are not linearly independent and $\det(C_I) = 0$.

Case 3: $\{f_1, \ldots, f_{n-1}\}$ incorporates all n vertices and is connected. The collection of edges forms a spanning tree. Clipping leaf vertices and edges, we can permute the rows and columns of C_I to produce a lower-triangular matrix with ± 1 in each diagonal entry. This implies $\det(C_I) = \pm 1$.

Example: $G = K_n$. The rows of C are indexed by $1, 2, \ldots, n-1$ and the columns are indexed by all (p, q) where $1 \le p < q \le n$. The (i, j)-entry of CC^T is

$$\sum_{1\le p<q\le n} C_{i,(p,q)} C_{j,(p,q)}.$$

When $i < j$ the only non-zero contribution is $C_{i,(i,j)}C_{j,(i,j)} = (-1)(1) = -1$ and when $i = j$ the only non-zero contributions are $C^2_{i,(i,q)}$ where $i < q$ and $C^2_{i,(p,i)}$ where $p < i$, for a total of $n-1$. Hence CC^T is the matrix with $n-1$ down the diagonal and -1 elsewhere. This has determinant n^{n-2}.

12. **Theorem:** Kruskal's Algorithm produces a minimum weight spanning tree.

Proof: First note that when Kruskal's algorithm terminates, all vertices have been incorporated by connectedness. So the result is a spanning tree. We will prove that $\{e_1, \ldots, e_k\}$ is a subset of a minimum weight spanning tree T_k for each k using an induction argument.

Base Case: Let T_0 be any minimum weight spanning tree. If includes e_1, then set $T_1 = T_0$. If it doesn't include e_1, the subgraph $T_0 + e_1$ contains a cycle of ≥ 3 edges. Delete any one of these cycle edges not equal to e_1. Call it t_0 to emphasize that it belongs to T_0. By minimality of $weight(e_1)$, $weight(T_0 + e_1 - t_0) \leq weight(T)$. Since $T_0 + e_1 - t_0$ consists of $n-1$ edges and encompasses all vertices, it is a tree. So in fact $T_0 + e_1 - t_0$ is a minimum weight spanning tree. We set $T_1 = T_0 + e_1 - t_0$.

Induction hypothesis: There exists a minimum weight spanning tree T_k that contains the edges $e_1, \ldots, e_k$.

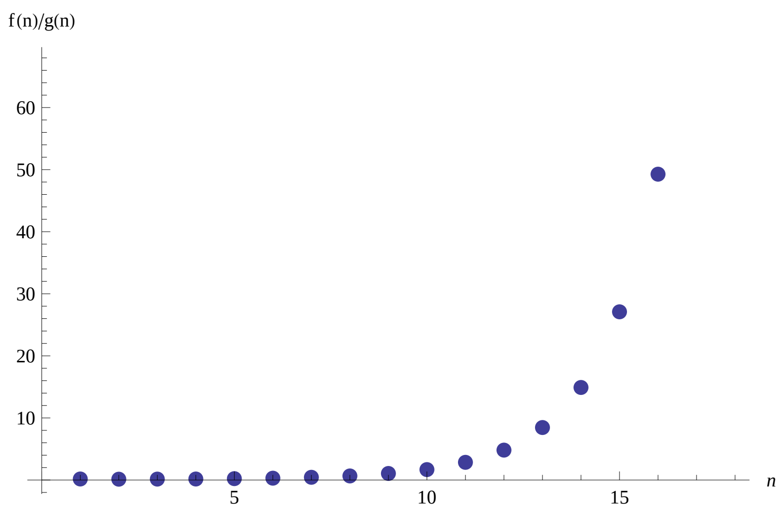
We must now construct a minimum weight spanning tree T_{k+1} that contains $e_1, \ldots, e_{k+1}$. If $e_{k+1} \in T_k$ then we set $T_{k+1} = T_k$. But if $e_{k+1} \notin T_k$ then $T_k + e_{k+1}$ contains a cycle. One of the edges in this cycle is e_{k+1}. Since the collection $\{e_1, \ldots, e_{k+1}\}$ is acyclic, one of the edges in the cycle cannot be in this set. Call it t_k to emphasize that it belongs to T_k. Then $T_k + e_{k+1} - t_k$ is a spanning tree. Since $\{e_1, \ldots, e_k, t_k\} \subseteq T_k$, the collection is acyclic. By the way e_{k+1} was chosen by Kruskal's Algorithm, $weight(e_{k+1}) \leq weight(t_k)$. Therefore $weight(T_k + e_{k+1} - t_k) \leq weight(T_k)$. Hence $T_k + e_{k+1} - t_k$ is a minimum weight spanning tree. We set $T_{k+1} = T_k + e_{k+1} - t_k$.

13. **Theorem:** Let G be a graph and let A be its adjacency matrix. Then the entries of A^n record the number of walks of length n between vertices.

Proof: Use induction and a counting argument.

Chapter 10 Exercises, p. 234: (1) We must $a_n = 1$, otherwise the sum is too large. Now discard a_n and replace a_{n-1} by $a_{n-1} - 1$. The sum is now $2n - 4$. Use an induction argument. (2) Each non-leaf accounts for

k edges, so there are km edges. Hence $km = e = v - 1$, $v = km + 1$. Hence there are $v - m = m(k-1) + 1$ leaf edges. (3) Using (1), have have at least $p(2n-2, n \text{ parts})$ distinct degree sequences. Hence at least $p(2n-2, \text{first part } n)$ trees. Hence at least $p(n-2)$ trees. (4) Let there be t_n non-isomorphic trees. Picking one of each class, we can generate all the others by applying vertex permutations. Hence there are at most $n!t_n$ trees. In other words, $n!t_n \geq n^{n-2}$. This says $t_n \geq \frac{n^{n-2}}{n!}$. Setting $f(n) = \frac{n^{n-2}}{n!}$ and $g(n) = p(n-2)$ and graphing $f(n)/g(n)$ versus n we can see that $f(n)$ is growing much more quickly than $g(n)$, hence $f(n)$ is the better lower bound:



(9) It suffices to prove that if G is not connected then G' is connected. Write $G = G_1 + G_2 + \cdots + G_k$, where $G_k \geq 2$ and the G_i are connected components of G. Then there is an edge in G' between every vertex of G_i and every vertex of G_j where $i \neq j$. Now consider two vertices x and y in G'. If they are in different G_i then they are joined by an edge in G'. But if they are both in some G_i, they are both joined to the same vertex in G_j where $j \neq i$, so there is a path of length 2 between x and y in G'. (10) Write $G = G_1 + G_2 + \cdots + G_k$, connected component decomposition. Assume that G_i has n_i vertices and that G has n vertices. Then G can have at most $\binom{n_1}{2} + \cdots + \binom{n_k}{2}$ edges. We wish to find the maximum value of such a sum. We can obviously increase the sum by adding edges to form a complete graph out of the first $k-1$ components and still have a disconnected graph. We are reduced to computing the maximum value of $\binom{a}{2} + \binom{b}{2}$ where $a + b = n$ and $a, b \geq 1$. If $a \geq b \geq 2$ then removing a vertex from the smaller component we

lose $b-1$ edges, and adding this vertex to the larger component and adding all possible edges we gain a edges, with a net gain of $a-b+1 \geq 1$ edges. Keep on going until we obtain $G = K_{n-1} + K_1$ having $\binom{n-1}{2}$ edges. (11) Let the connected component decomposition of G be $G_1 + \cdots + G_k$ and let a corresponding forest be $F_1 + \cdots + F_k$. This forest satisfies $v(F) - e(F) = k$. Therefore $e(F) = v(F) - k = n - k$, which implies $e(G) - e(F) = m - n + k$. Every edge in $E(G) - F(G)$ that we add to F creates a cycle, so there are at least $m-n+k$ cycles in the graph. (15) (a) This says their is a walk of length 4 from a vertex to itself, which is true. (b) There is a closed walk of length 11 from i to i. So we just have to show that a closed walk of odd length produces a cycle of odd length. Proceed along the walk until the first time you meet a vertex for the second time. Between the first and second time is a cycle. If odd, great. If even, chop it out and deal with a smaller closed walk of odd length. Keep on going. (c) The shortest walk between i and j, if there is one, has length n. Chopping out cycles we find a path of length n, which is too long since there are only n vertices. So there are no walks at all between i and j. (16) Walks of even length join two vertices on the same side only, and walks of odd length join two vertices on opposite sides. This gives information about the entries in A^m. (19) To create a forest, form an unordered set partition and place a tree structure on each part. (20) To create a rooted tree, create an ordered set partition where the first part has size 1 only and the second one is used to create a forest. To create a forest, form an unordered set partition and place a rooted tree structure on each part.

Supplementary Exercises, p. 237: (21) A forest creates a set partition. (23) These are path graphs. Choose the two endpoints in $\binom{n}{2}$ ways, then choose the order of $n-2$ vertices from smaller endpoint to larger endpoint in $(n-2)!$ ways. Total: $\frac{n!}{2}$. (24) Use a brute-force argument. (25) Keep track of the number of vertices in each tree. (26) If they don't, draw two parallel paths. Connected forces a path between them. Find a longer path in this structure. (30) Join vertices 1 through $n-1$ to vertex n. Then any subset of vertices that includes n will have induced subgraph which is a tree. There are 2^{n-1} of these. (31) I'm guessing 7: a vertex of degree 3 with paths of lengths 1, 2, and 3 away from it. (33) Just do it. (34)–(45) skip.

Chapters 12 and 18: Planar Graphs, Counting Unlabeled Structures (11 meetings)

Planar Graphs concepts: Planar graph, circle-chord method for proving/disproving planarity, Kuratowski's theorem, planar graph regions, Euler's Formula, dot-counting argument, Euler's inequality, numerical methods for disproving planarity, polyhedron, the five regular polyhedra, the five-color theorem.

Planar graph: Can be drawn in plane with no crossing edges.

K_5 is non-planar using circle-chord method. K_{33} is non-planar.

Kuratowski's Theorem: a graph is planar if and only if it does not have a subgraph which can be described as a subdivision of K_5 or K_{33}.

Example: the Petersen graph has a K_{33} configuration, so it is non-planar.

Planar graphs define regions. It turns out that all possible planar representations of a connected graph create the same number of regions.

Lemma 1: A planar graph containg a cycle can be redrawn so that the cycle bounds the remaining vertices and edges and the number of regions does not change.

Proof: Find a cycle of minimal size. There will be no chords. Stuff everything inside this cycle (imagine drawing the image on a rubber ball, then puncturing the ball inside the cycle then tearing and spreading out flat. Things will deform but the number of regions will remain the same).

Theorem 2: For a connected planar graph, $r = e - v + 2$. (Note: r is f in this textbook (face).)

Proof: If the graph is a tree, there is one region, which satisfies the formula. If the graph is not a tree, there is a cycle somewhere. Stuff everything inside a minimal cycle as in Lemma 1. Delete one of the edges of the outer cycle. In the process, lose one interior region and lose one edge. Keep on going until you obtain a tree. We have $r' = e' - v + 2$ in the tree, hence $r = e - v + 2$.

Example: two different representations of K_4.

Theorem 3: If a connected planar graph contains a cycle then $e \leq 3v - 6$.

Proof: Stuff everything inside the cycle as before. Now place 2 dots about every edge near the middle. The dots are partitioned by region, and each region is bounded by a cycle. So each region contributes at least 3 dots. So the number of dots is $\geq 3r$. On the other hand, the number of dots is $2e$, therefore $2e \geq 3r$. That is, $2e \geq 3e - 3v + 6$. This yields $e \leq 3(v - 2)$.

Application: K_5 has $e = 10$ and $3v - 6 = 9$, therefore it is not planar.

Note: K_{33} is non-planar, yet $e = 9$ and $3v - 6 = 12$, so it is one of the many non-planar graphs that also satisfy this inequality.

Theorem 4: If a connected planar graph contains a cycle and every cycle has at least k edges then $e \leq \frac{k}{k-2}(v-2)$.

Proof: We can replace $2e \geq 3r$ by kr. Hence $2e \geq k(e-v+2)$. Rearranging, $e \leq \frac{k}{k-2}(v-2)$.

Application: K_{33} satisfies $k = 4$. If it is planar it must satisfy $e \leq 2(v-2)$. But it doesn't: $e = 9$, $2(v-2) = 8$.

Application: the Petersen graph satisfies $k = 5$. If it is planar it must satisfy $e \leq \frac{5}{3}(v-2)$. But it doesn't: $e = 15$, $\frac{5}{3}(v-2) = \frac{40}{3} = 13\frac{1}{3}$.

Polyhedron: a solid whose boundary is a union of polygons. Can be associated with a planar graph by projecting onto surface of sphere, then puncturing the sphere and laying out flat.

Example: the cube. Draw as one square inside another, with corresponding vertices joined by edges. Can be identified with Q_3.

Regular polyhedron: all vertices have the same degree and every region has the same number of edges.

Theorem 5: There are exactly 5 regular polyhedra. (p. 287: tetrahedron, cube, dodecahedron, octahedron, icosahedron.)

Proof: To prove that there are no others, note that if degrees are d and bounding cycle lengths are k then $2e = vd = rk$ by a dot counting argument. Combined with $r - e + v = 2$ we obtain

$$2\frac{e}{k} - e + 2\frac{e}{d} = 2$$

$$\frac{2}{k} - 1 + \frac{2}{d} = \frac{2}{e}$$

$$\frac{2}{k} - 1 + \frac{2}{d} > 0$$

$$\frac{2}{k} + \frac{2}{d} > 1$$

$$2d + 2k > dk$$

$$(d-2)(k-2) = dk - 2d - 2k + 4 < 4.$$

Since $d \geq 3$ and $k \geq 3$ the only solutions to this inequality are

$$(d,k) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}.$$

Theorem 6: Every planar graph contains a vertex of degree ≤ 5.

Proof: We can assume without loss of generality that the graph is connected. It cannot be a tree since a tree has at least one degree ≤ 1 vertex. Hence the graph must satisfy $e \leq 3v - 6$. If all vertex degrees are ≥ 6 then the vertex-degree sum yields $2e \geq 6v$ or $e \geq 3v$: contradiction. So there has to be a degree ≤ 5 vertex.

Theorem 7: Every planar graph is 5-colorable.

Proof: By induction on number of vertices. If there is one vertex, fine. If there is more than one vertex, pick one (v) of degree ≤ 5 and delete it. The resulting graph $G - v$ is planar and 5-colorable. If the degree of v is 4 or less, we can assign v a color to avoid a color-conflict. If the degree of v is 5 but is only joined to 4 or less colors, then we can still assign v a color to avoid a color-conflict. In the worst-case scenario, v is joined to vertices of colors 1 through 5. In this case we will modify the colors in $G - v$ first so that v is joined to at most 4 different colors.

How to do this: imagine that v is joined to vertices v_1, v_2, v_3, v_4, v_5 colored 1,2,3,4,5 (reading the vertices clockwise). Let $G[\{1,3\}]$ be the subgraph induced by vertices colored 1 and 3. No color conflicts result by swapping the colors 1 and 3 in any connected component of $G[\{1,3\}]$ and leaving the other colors alone. Similarly, let $G[\{2,4\}]$, the subgraph induced by vertices colored 1 and 3. No color conflicts result by swapping the colors 2 and 4 in any connected component of $G[\{2,4\}]$ and leaving the colors alone. So these are safe color modifications in $G - v$.

Case 1: v_1 and v_3 are in the same connected component of $G[\{1,3\}]$. So there is a path from v_1 to v_3 in which all vertex colors are 1 and 3. Since every path from v_2 to v_4 has to intersect a vertex in this path from v_2 to v_4 by planarity, v_2 and v_4 cannot be in the same connected component of $G[\{2,4\}]$. Swap the colors 2 and 4 in the component of $G[\{2,4\}]$ containing v_2. Then v is joined to colors 1,3,4,5 and can be safely colored 2.

Case 2: v_1 and v_3 are not in the same connected component of $G[\{1,3\}]$. Swap the colors 1 and 3 in the component of $G[\{1,3\}]$ containing v_1. Then v is joined to the colors 2, 3, 4, 5 and can be safely colored 1.

Chapter 12 Exercises, p. 290: (1) Adapt the proof given. In the case of a forest, we know that there are $v-e$ trees and exactly one region, so we have $r = e-v+k+1$ where k is the number of connected components. More generally, starting with a graph with k connected components, stuff each inside a cycle and cut a bounding cycle edge. Lose an edge and lose a region. Keep on going until down to a forest, satisfying $r' = e'-v+k+1$. This implies $r = e-v+k+1$. (2) We did this by associating the polyhedron with a planar graph. (3) To satisfy $3r = 2e$, every region must be bounded by exactly 3 edges. This implies $e = 3v-6$ using Euler's formula. In a polyhedron, each vertex degree is at least 3, so $2e \geq 3v$. Combining $e = 3v-6$ with $2e = 3v$ yields $e = 6$, $v = 4$. So the graph in question is K_4, drawn like a triangle with central vertex joined to each of the three vertices of the triangle. (4) The degrees in any graph are in the range 0 through $v-1$. But in polyhedron, there degrees are ≥ 3, so two of the degrees must be the same by the Pigeonhole principle. (5) Create the dual of the polyhedron, which has vertex set consisting of faces and edge set consisting of pairs of neighboring faces. We get another polyhedron, so we can apply (4). To illustrate, do this to the cube. (6) We just did. (7) skip (8) skip (9) skip (10) In a bipartite graph with a cycle, cycle lengths are ≥ 4, so using Theorem 4, $e \leq 2(v-2)$. On the other hand, by the vertex degree sum $2e \geq dv$. Hence $dv \leq 2e \leq 4(v-2)$. This rearranges to $(4-d)v \geq 8$. This forces $d \leq 3$. We can obtain $d = 3$ using Q_3.

Chapter 12 Supplementary Exercises, p. 290: (11) There exist planar graphs with all vertices of degree 5. (17) No: K_{33}. (18) $K_6 - e_1 - e_2$ violates $e \leq 3v-6$. There is a planar $K_6 - e_1 - e_2 - e_3$: triangulate C_6 both inside and outside. (19) I'm assuming $a \neq b$. On the one hand, $2e = 3n$ by the vertex-sum theorem. On the other hand, $2e = ap_a + bp_b$ by a dot-counting argument. So we have $3n = ap_a + bp_b$. A third relation we can apply is $r = e-v+2$, which yields $p_a + p_b = \frac{3}{2}n - n + 2 = \frac{n}{2} + 2$. So we have $(6-a)p_a + (6-b)p_b = 6(p_a+p_b) - (ap_a + bp_b) = (3n+12) - (3n) = 12$. (20) No, because $a < b$ is required. Do all of the latter exist? We can calculate $(6-a)p_a + (6-b)p_b$ by brute force and see when it equals 12. The number of vertices is $\frac{1}{3}(ap_a + bp_b)$. $(3,4)$ exists using $p_3 = 2$ and $p_4 = 3$ (see figure).

$(4,5)$ exists using $p_4 = 5$ and $p_5 = 2$ (see figure). $(3,5)$: haven't been able to do it so far.

Digraph Model of Equivalent Colorings

Review of graph automorphisms: vertex permutations that preserve graph structure. Composition, identity, inverses, cancellation law.

There are two types of graph colorings: vertex colorings and edge colorings. We will start with vertex colorings.

Let G be a graph with vertex set $[n]$. A k-coloring of the vertices of a graph G is simply a function $c : [n] \to [k]$, which can be viewed as an assignment of one color from $[k]$ to each vertex of G. A counting argument shows that there are k^n k-colorings of G.

Now let D_k be the directed graph with vertex set consisting of k-colorings of G and a directed edge of the form $c \to_\sigma d$ whenever c and d are colorings, σ is a graph automorphism of G, and permuting the vertices in G using σ and leaving the color labels from c in place produces the coloring d. This can be expressed symbolically as $c \circ \sigma = d$.

We will say that $c \equiv d$ if and only if $c \to_\sigma d$ for some $\sigma \in \mathrm{aut}(G)$. Viewing $c \equiv d$ as a process of permuting vertices while leaving color labels in place, it is easy to see that $c \equiv c$, that $c \equiv d$ implies $d \equiv c$, and that $c \equiv d$ and $d \equiv e$ implies $c \equiv e$. Hence $\equiv$ is an equivalence relation. If c and d are not equivalent, we will call them inequivalent.

Observations about the structure of D_k:

1. D_k has k^n vertices and $|\mathrm{aut}(G)|k^n$ edges.

2. There is a directed edge of the form $c \to_\sigma d$ if and only if c and d are equivalent, so the connected components of D_k are strongly connected with at least one directed edge connecting every pair of vertices, including a vertex to itself. All colorings in a given component are equivalent to each other. The number of inequivalent colorings of G is defined to be m, the number of strongly connected components of D_k.

3. Loop edges are edges of the form $c \to_\sigma c$. There is at least one loop edge at each vertex, using the identity automorphism. We will prove that there are $m|\mathrm{aut}(G)|$ loop edges in D.

Lemma: Let c, d, and e be equivalent colorings of G. Then

$$|c \to d| = |c \to e|,$$

where $x \to y$ is notation for the set of all edges of the form $x \to_\sigma y$ in D_k.

Proof: Since $d \equiv e$, there is a graph automorphism τ such that $d \to_\tau e$. This gives rise to a function from $c \to d$ to $c \to d$ via

$$f(c \to_\sigma d) = c \to_{\tau\sigma} e.$$

The function f is surjective, because if $c \to_\sigma e$ then $c \to_{\tau^{-1}\sigma} d$ and

$$f(c \to_{\tau^{-1}\sigma} d) = f(c \to_\sigma e).$$

Therefore $|c \to d| \geq |c \to e|$. Reversing the roles of d and e, we also have $|c \to e| \geq |c \to d|$. Therefore $|c \to d| = |c \to e|$.

Corollary: For each $c \in D_k$, $|c \to c| = \frac{|\mathrm{aut}(G)|}{|V(C)|}$, where C is the strongly connected component containing c.

Proof: Since the number of edges from c to d is equal to the number of edges from c to c for every $d \in C$, the total number of edges out of c is $|V(C)| \times |c \to c|$. On the other hand, we know that the total number of edges out of c is $|\mathrm{aut}(G)|$, hence

$$|V(C)| \times |c \to c| = |\mathrm{aut}(G)|,$$

hence the formula.

Corollary: Let C be a connected component of D_k. Then C has $|\mathrm{aut}(G)|$ loop edges.

Proof: The number of loop edges in C is

$$\sum_{c \in V(C)} |c \to c| = \sum_{c \in V(C)} \frac{|\mathrm{aut}(G)|}{|V(C)|} = |V(C)| \frac{|\mathrm{aut}(G)|}{|V(C)|} = |\mathrm{aut}(G)|.$$

Corollary: D_k has $m|\mathrm{aut}(G)|$ loop edges.

4. For each $\sigma \in \mathrm{aut}(G)$ let $\mathrm{FixedBy}(\sigma)$ be the set of colorings c that satisfy $c \to_\sigma c \in D$. Then

$$m = \frac{1}{|\mathrm{aut}(G)|} \sum_{\sigma \in \mathrm{aut}(G)} |\mathrm{FixedBy}(\sigma)|.$$

Proof: Each $\sigma \in \mathrm{aut}(G)$ contributes $|\mathrm{FixedBy}(\sigma)|$ loop edges. Since the total number of loop edges in D_k is $m|\mathrm{aut}(G)|$, we obtain

$$\sum_{\sigma \in \mathrm{aut}(G)} |\mathrm{FixedBy}(\sigma)| = m|\mathrm{aut}(G)|.$$

Dividing by $|\mathrm{aut}(G)|$ we obtain the formula.

5. For each $\sigma \in \mathrm{aut}(G)$,

$$|\mathrm{FixedBy}(\sigma)| = k^{\#\text{ cycles in }\sigma}.$$

Proof: If a cycle in σ is $(x_1, x_2, x_3, \ldots)$, and σ fixes the coloring c, then the color of x_1 has to be the color of x_2, which has to be the color of x_3, etc. In other words, all the vertices in the cycle have to be assigned the same color by c. So we obtain all possible colorings fixed by σ by assigning one color per cycle of σ in all possible ways. If σ has p cycles in it then the number of k-colorings of G fixed by σ is equal to k^p.

6. Combining (4) and (5), we obtain

$$m = \frac{1}{|\mathrm{aut}(G)|} \sum_{\sigma \in \mathrm{aut}(G)} k^{\#\text{ cycles in }\sigma}.$$

7. If we restrict ourselves to a subgroup H of $\mathrm{aut}(G)$, then all the same arguments above apply and we obtain

$$m = \frac{1}{|H|} \sum_{\sigma \in H} k^{\#\text{ cycles in }\sigma}.$$

8. If we restrict ourselves to colorings with n_1 vertices of color 1, n_2 vertices of color 2, ..., n_k vertices of color k, then all the same arguments above apply and we obtain

$$m(n_1, n_2, \ldots, n_k) = \frac{1}{|H|} \sum_{\sigma \in H} |\mathrm{FixedBy}(\sigma)|.$$

But the formula for FixedBy(σ) has to be adjusted.

Let σ be permutation with x_1 1-cycles, x_2 2-cycles, ..., x_n n-cycles. Place a z_i underneath each vertex colored i. Under the 1-cycles we have a sequence of x_1 variables. Under the 2-cycles we have a sequence of x_2 squared variables. ... Under the n-cycles we have a sequence of x_n variables raised to the n^{th} power. The product of these variables is $z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}$. We can count these sequences by expanding the product

$$(z + 1 + z_2 + \cdots + z_k)^{x_1}(z_1^2 + z_2^2 + \cdots + z_k^2)^{x_2} \cdots (z_1^n + z_2^n + \cdots + z_k^n)^{x_n}$$

and taking the coefficient of $z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}$. This yields

$$|\text{FixedBy}(\sigma)| = [z_1^{n_1} \cdots z_k^{n_k}] Z_1^{\#\text{1-cycles of } \sigma} Z_2^{\#\text{2-cycles of } \sigma} \cdots Z_k^{\#k\text{-cycles of } \sigma}$$

where

$$Z_i = z_1^i + z_2^i + \cdots + z_k^i.$$

Hence

$$m(n_1, n_2, \ldots, n_k) =$$
$$[z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}] \frac{1}{|H|} \sum_{\sigma \in H} Z_1^{\#\text{1-cycles of } \sigma} Z_2^{\#\text{2-cycles of } \sigma} \cdots Z_k^{\#k\text{-cycles of } \sigma}.$$

A generating for all these numbers is the polynomial

$$\frac{1}{|H|} \sum_{\sigma \in H} Z_1^{\#\text{1-cycles of } \sigma} Z_2^{\#\text{2-cycles of } \sigma} \cdots Z_k^{\#k\text{-cycles of } \sigma}.$$