Power Series Methods

1. Let

$$y = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Then

$$\frac{d^k y}{dx^k} = \sum_{n=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)a_n}{n!} x^{n-k} = \sum_{n=k}^{\infty} \frac{a_n}{(n-k)!} x^{n-k} = \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!} x^n.$$

Also,

$$x^k y = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^{n+k} = \sum_{n=k}^{\infty} \frac{a_{n-k}}{(n-k)!} x^n = \sum_{n=0}^{\infty} \frac{a_{n-k}}{(n-k)!} x^n$$

if we insist that $a_{-1} = a_{-2} = \cdots = 0$. Combining these two operations, we get the general formula

$$y = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \qquad x^p \frac{d^q y}{dx^q} = \sum_{n=0}^{\infty} \frac{a_{n+q-p}}{(n-p)!} x^n.$$

2. Example: consider the differential equation

$$y'' - (x^3 + 2)y' - 6x^2 y = 0, \qquad y(0) = y'(0) = 1.$$

Then $x_0 = 0$ is an ordinary point, and we can expand in powers of x. Setting $y = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$, we see that $a_0 = a_1 = 1$. Substituting

$$y = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

into the differential equation, we get

$$\sum_{n=0}^{\infty} \frac{a_{n+2}}{n!} x^n - \sum_{n=0}^{\infty} \frac{a_{n+1-3}}{(n-3)!} x^n - 2\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n - 6\sum_{n=0}^{\infty} \frac{a_{n-2}}{(n-2)!} x^n = 0.$$

Therefore

$$\frac{a_{n+2}}{n!} - \frac{a_{n-2}}{(n-3)!} - 2\frac{a_{n+1}}{n!} - 6\frac{a_{n-2}}{(n-2)!} = 0.$$

Multiplying through by $n!$ we obtain

$$a_{n+2} - n(n-1)(n-2)a_{n-2} - 2a_{n+1} - 6n(n-1)a_{n-2} = 0.$$

Therefore

$$a_{n+2} = 2a_{n+1} + n(n-1)(n+4)a_{n-2} \qquad n \geq 0.$$

Hence

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 1\\
a_2 &= 2a_1 = 2\\
a_3 &= 2a_2 = 4\\
a_4 &= 2a_3 + 6a_0 = 14\\
a_5 &= 2a_4 + 42a_1 = 50\\
&\dots
\end{aligned}$$

$$y = \frac{1}{0!} + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{50}{5!}x^5 + \cdots.$$

3. Next consider the differential equation

$$2x^2y'' + 3xy' + (4x - 6)y = 0.$$

Then $x_0 = 0$ is a regular singular point, and we use the method of Frobenius. We will first make the change of variables $y = x^r z$, then determine r using the indicial equation. Then we solve for $z = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$.

We have

$$y' = (x^r z)' = rx^{r-1}z + x^r z'$$

and

$$y'' = (x^r z)'' = r(r-1)x^{r-2}z + 2rx^{r-1}z' + x^r z'',$$

therefore

$$2x^2(r(r-1)x^{r-2}z + 2rx^{r-1}z' + x^r z'') + 3x(rx^{r-1}z + x^r z') + (4x - 6)x^r z = 0.$$

Cleaning this up and factoring we obtain

$$x^r[2xz'' + (4r + 3)xz' + (4x + 2r^2 + r - 6)z] = 0.$$

If we expand z as a power series as above, then the lowest power of x in this equation is $(2r^2 + r - 6)x^r$, which we want to be equal to zero. Therefore the indicial equation

$$2r^2 + r - 6 = 0$$

determines two solutions for r, namely $r = -2$ and $r = \frac{3}{2}$. Using $r = -2$ we obtain

$$x^{-2}[2xz'' - 5xz' + 4xz = 0.$$

Therefore

$$2z'' - 5z' + 4z = 0,$$

which by some miracle is a constant coefficients equation (this is not always the case). We can determine the coefficients for z as above, or we can solve this constant coefficients equation by the method of guessing $z = e^{mx}$. Choosing the former method, we obtain

$$2\sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}x^n - 5\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}x^n + 4\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n = 0,$$

which implies

$$2\frac{a_{n+2}}{n!} - 5\frac{a_{n+1}}{n!} + 4\frac{a_n}{n!} = 0.$$

Therefore

$$a_{n+2} = \frac{5a_{n+1} - 4a_n}{2}, \qquad n \geq 0.$$

Hence

$$\begin{aligned} a_2 &= -2a_0 + 2.5a_1 \\ a_3 &= -5.0a_0 + 4.25a_1 \\ a_4 &= -8.5a_0 + 5.265a_1 \\ \cdots \end{aligned}.$$

Hence

$$z = \frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{-2a_0 + 2.5a_1}{2!}x^2 + \frac{-5.0a_0 + 4.25a_1}{3!}x^3 + \frac{-9.5a_0 + 5.265a_1}{4!}x^4 + \cdots =$$

$$a_0(1 - \frac{2}{2!}x^2 - \frac{5}{3!}x^3 - \frac{9.5a_0}{4!}x^4 + \cdots) +$$
$$a_1(\frac{1}{1!}x + \frac{2.5}{2!}x^2 + \frac{4.25}{3!}x^3 + \frac{5.265}{4!}x^4 + \cdots).$$

Since this z is a linear combination of two linearly independent functions (use Wronksian), this is the general solution for z. Hence the general solution for y is

$$y = x^r z =$$

$$a_0x^{-2}(1 - \frac{2}{2!}x^2 - \frac{5}{3!}x^3 - \frac{9.5a_0}{4!}x^4 + \cdots) +$$
$$a_1x^{-2}(\frac{1}{1!}x + \frac{2.5}{2!}x^2 + \frac{4.25}{3!}x^3 + \frac{5.265}{4!}x^4 + \cdots).$$