# Differential Equation Notes

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## 1 Trajectories

A trajectory in $\mathbf{R}^n$ is a function $\alpha : [t_0, t_1] \to \mathbf{R}^n$ of the form

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)).$$

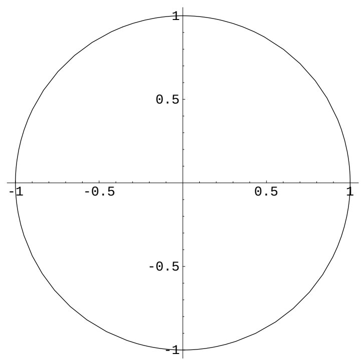
The graph of the trajectory is

$$\{\alpha(t) : t \in [t_0, t_1]\}.$$

**Example 1.1.** Let $\alpha : [0, 2\pi] \to \mathbf{R}^2$ be defined by

$$\alpha(t) = (\cos t, \sin t).$$

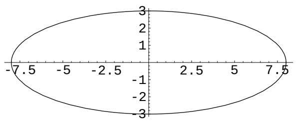
The graph of $\alpha$ is $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$, the unit circle.



**Example 1.2.** Let $\alpha : [0, 2\pi] \to \mathbf{R}^2$ be defined by

$$\alpha(t) = (8\cos t, 3\sin t).$$

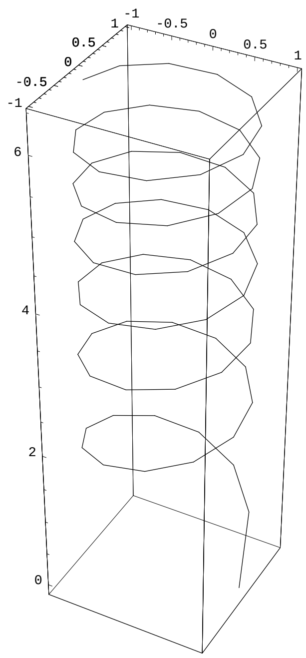
The graph of $\alpha$ is $\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{8^2} + \frac{y^2}{3^2} = 1\}$, the ellipse with major axis of length 16 and minor axis of length 6.




**Example 1.3.** Let $\alpha : [0, 13\pi] \to \mathbf{R}^3$ be defined by

$$\alpha(t) = (2\cos t, 2\sin t, \sqrt{t}).$$

The graph of $\alpha$ is a helix of radius 1. As $t$ increases the graph becomes increasingly compressed.



## 2 Direction vectors

Each point $\alpha(t)$ of a trajectory $\alpha$ can be regarded as a vector which begins at the origin and ends at $\alpha(t)$. The displacement from position $\alpha(t)$ to $\alpha(t')$ is the vector $\alpha(t+h) - \alpha(t)$. The average rate of change of position from time $t$ to time $t+h$ is

$$\frac{1}{h}(\alpha(t+h) - \alpha(t)).$$

The instantaneous rate of change of the position vector at time $t$ is the limit

$$\alpha'(t) = \lim_{h\to 0} \frac{1}{h}(\alpha(t+h) - \alpha(t)).$$

If

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)),$$

then

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \ldots, \alpha_n'(t)).$$

We can interpret $\alpha'(t)$ as the direction a particle is heading in at time $t$ as it is traveling along the trajectory $\alpha$. If $\alpha(t) = (x(t), y(t))$ then the slope of the direction vector at time $t$ is $\frac{y'(t)}{x'(t)}$, assuming $x'(t) \neq 0$.

**Example 2.1.** In Example 1.1, $\alpha'(t) = (-\sin t, \cos t)$. At time $t = \frac{\pi}{4}$, the particle is in position $\alpha(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and is heading in direction

$$\alpha'(\frac{\pi}{4}) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

with slope $-1$.

**Example 2.2.** In Example 1.2, $\alpha'(t) = (-8\sin t, 3\cos t)$. At time $t = \frac{\pi}{4}$, the particle is in position $\alpha(\frac{\pi}{4}) = (8\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2})$ and is heading in direction

$$\alpha'(\frac{\pi}{4}) = (-8\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2})$$

with slope $-\frac{3}{8}$.

**Example 2.3.** In Example 1.2, $\alpha'(t) = (-\sin t, \cos \frac{1}{\sqrt{t}})$. At time $t = k\pi$, $1 \leq k \leq 12$, the particle is in position $\alpha(k\pi) = (-1, 0, \sqrt{k\pi})$ and is heading in direction $\alpha'(k\pi) = (0, -1, \frac{2}{\sqrt{k\pi}})$. Notice that the direction vectors are becoming more horizontal as $t$ increases.

**Example 2.4.** In Examples 1.1 and 1.2, both trajectories travel exactly once around their graphs in the counter-clockwise direction. At what times are both particles traveling in the same direction?

Answer: at those times $t$ in which the direction vectors are parallel to each other, namely when

$$(-\sin t, \cos t) = \lambda(t)(-8\sin t, 3\cos t)$$

for some $\lambda(t) \neq 0$. When $\sin t \neq 0$ then we must have $\lambda(t) = \frac{1}{8}$, which forces $\cos t = \frac{3}{8}\cos t$, which forces $\cos t = 0$, which forces $\sin t = \pm 1$. This corresponds to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. When $\sin t = 0$ we must have $\cos t = \pm 1$ and $\lambda(t) = \frac{1}{3}$. This corresponds to $t = 0$ and $t = \pi$.

**Exercise 1:** Let $P$ be a particle traveling clockwise around the circle $x^2 + y^2 = 25$.

(a) Find the direction $P$ is traveling in at the moment it passes through the point $(-3, -4)$.

(b) Find both points along the circle at which the particle is traveling in a direction which is parallel to the line $y = 2x$. Hint: the line has direction vector $(1, 2)$.

**Exercise 2:** Let $P$ be a particle traveling clockwise around the ellipse $\frac{x^2}{25} + \frac{y^2}{100} = 1$.

(a) Find the direction $P$ is traveling in at the moment it passes through the point $(3, 8)$.

(b) Find both points along the ellipse at which the particle is traveling in a direction which is parallel to the line $y = 2x$.

# 3 Surfaces

Let $f : \mathbf{R^n} \to \mathbf{R}$ be given. A surface is the set of solutions to

$$f(x_1, x_2, \dots, x_n) = 0.$$

**Example 3.1.** Let $f(x, y) = x^2 + y^2 - 1$. The surface associated with $f$ is the unit circle (see Example 1.1).

**Example 3.2.** Let $f(x, y) = \frac{x^2}{8^2} + \frac{y^2}{3^2} - 1$. The surface associated with $f$ is an ellipse (see Example 1.2).

**Example 3.3.** Let $f(x, y, t) = (x - \cos t^2)^2 + (y - \sin t^2)^2$. The surface associated with $f$ is

$$\{(\cos t^2, \sin t^2, t) : t \in \mathbf{R}\} =$$
$$\{(\cos t, \sin t, \sqrt{t}) : t \in \mathbf{R}\} \cup \{(\cos t, \sin t, -\sqrt{t}) : t \in \mathbf{R}\}.$$

This is a helix. Compare with Example 1.3.

# 4 Partial derivatives and the chain rule

Let $f(x, y, z) = y - 2zx + \frac{2}{3}z^3$. The partial derivatives of $f$ are

$$\frac{\partial f}{\partial x}(x, y, z) = -2z, \ \frac{\partial f}{\partial y}(x, y, z) = 1, \ \frac{\partial f}{\partial z}(x, y, z) = -2x + 2z^2.$$

If we assume that $x$, $y$, and $z$ are functions of $t$, then we can define

$$F(t) = f(x(t), y(t), z(t)) = y(t) - 2z(t)x(t) + \frac{2}{3}z(t)^3.$$

The derivative is

$$F'(t) = y'(t) - 2z'(t)x(t) - 2z(t)x'(t) + 2z(t)^2 z'(t) =$$

$$-2z(t) \cdot x'(t) + 1 \cdot y'(t) + (-2x(t) + 2z(t)^2) \cdot x'(t) =$$

$$\left[\frac{\partial f}{\partial x}(x(t), y(t), z(t))\right] x'(t) + \left[\frac{\partial f}{\partial x}(x(t), y(t), z(t))\right] y'(t) + \left[\frac{\partial f}{\partial x}(x(t), y(t), z(t))\right] z'(t).$$

In short,

$$\frac{d}{dt} f = f_x x' + f_y y' + f_z z'.$$

# 5 Calculating direction vectors on a curve defined as a surface

Let $\alpha(t) = (x(t), y(t))$ be a trajectory which traces out the curve defined by the surface

$$f(x, y) = 0.$$

Then the direction of the trajectory at time $t$ is the vector $(x'(t), y'(t))$. On the other hand, we know that

$$F(t) = f(x(t), y(t)) = 0$$

for all $t$, therefore

$$F'(t) = f_x x' + f_y y' = 0$$

for all $t$, therefore $(x', y')$ is perpendicular to $(f_x, f_y)$ and $(x', y')$ is parallel to $(-f_y, f_x)$.

**Example 5.1.** Let $f(x, y) = x^2 + y^2 - 1$. The direction of any trajectory along the surface generated by $f$ is parallel to

$$(-f_y, f_x) = (-2y, 2x)$$

with slope $-\frac{x}{y}$ at the point $(x, y)$. In particular, the slope of the direction vector of a particle traveling around the unit circle $x^2 + y^2 = 1$ at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is $-1$. Compare this with Example 2.1.

**Example 5.2.** Let $f(x, y) = \frac{x^2}{8^2} + \frac{y^2}{3^2} - 1$. The direction of any trajectory along the surface generated by $f$ is parallel to

$$(-f_y, f_x) = (-\frac{2}{9}y, \frac{2}{64}x)$$

with slope $\frac{-9x}{64y}$ at the point $(x, y)$. In particular, the slope of the direction vector of a particle traveling around the ellipse $\frac{x^2}{8^2} + \frac{y^2}{3^2} = 1$ at the point $(8\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2})$ is $\frac{-72}{192} = \frac{-3}{8}$. Compare this with Example 2.2.

# 6 1-parameter families

The function $f : \mathbf{R}^3 \to \mathbf{R}$ can be used to define a one-parameter family of curves. The curve corresponding to $z = c$ is the graph of all $(x, y)$ satisfying

$$f(x, y, c) = 0.$$

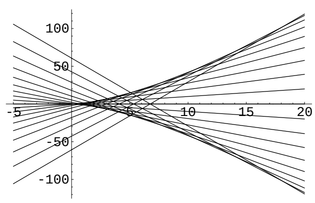
**Example 6.1.** $f(x, y, z) = y - 2zx + \frac{2}{3}z^3$. For $c = 1$ we get

$$y = 2x - \frac{2}{3}.$$

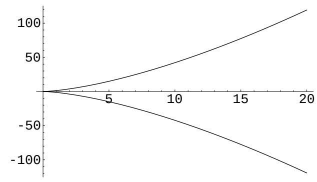
For $c = 2$ we get

$$y = 4x - \frac{16}{3}.$$

All the curves in this family are straight lines. If we plot the lines corresponding to $c = 0.5k$ for $k \in \{-10, -9, \ldots, 9, 10\}$ then we obtain



There seems to be a curve which is tangent to all these lines. This is called the envelope of the 1-parameter family. The envelope is




# 7 Calculating the envelope of a 1-parameter family

Given a 1-parameter family of curves defined by $f(x, y, z) = 0$, we can calculate the envelope (if it exists) as follows: Let $\alpha(t) = (x(t), y(t))$ be a parameterized curve which is a candidate for the envelope. Properties $\alpha$ must have:

1. It must pass through every curve defined by the equation $f(x, y, c) = 0$ for each $c$. Therefore, for each $t$ there must be a number $c(t)$ such that $f(x(t), y(t), c(t)) = 0$. We can say that $\alpha$ passes through the curve defined by $f(x, y, c(t)) = 0$ at time $t$.

2. We will make the assumption that $c(t)$ is a differentiable function of $t$ and that the partial derivatives of $f$ can be computed at all points of $\alpha$. Since $f(x(t), y(t), c(t))$ is a constant function of $t$, it must have time derivative equal to 0. Therefore

$$f_x x'(t) + f_y y'(t) + f_z c'(t) = 0.$$

3. Given a fixed time value $t$, we will define $F^{(t)}(x, y) = f(x, y, c(t))$. Any trajectory tracing out the curve defined by $F^{(t)}$ must have direction vector parallel to $(-F_y^{(t)}, F_x^{(t)}) = (-f_y, f_x)$ at the point $(x(t), y(t))$.

4. We want $\alpha(t)$ to be tangent to each curve in the 1-parameter family defined by $f$. Therefore we want $(x'(t), y'(t))$ to be parallel to the vector $(-f_y, f_x)$ for all $t$. Hence there must be a number $\lambda(t) \neq 0$ such that

$$(x'(t), y'(t)) = \lambda(t)(-f_y, f_x)$$

at time $t$.

5. Combining Properties 2 and 4, we want $f_z c'(t) = 0$. So to find $\alpha(t)$ we will solve the simultaneous equations $f(x, y, z) = 0$ and $f_z(x, y, z) = 0$, eliminate $z$ from this if possible, then parameterize the solutions $(x, y)$ with respect to a time variable $t$. We must then check that we can define a differentiable function $c(t)$ as in Properties 1 and 2.

**Example 7.1.** Consider $f(x, y, z) = y - 2zx + \frac{2}{3}z^3$ as in Example 6.1. Then we must solve

$$\begin{aligned} y - 2zx + \frac{2}{3}z^3 &= 0 \\ -2x + 2z^2 &= 0. \end{aligned}$$

We can see that $x = z^2$, therefore $y = \frac{4}{3}z^3$, therefore

$$y^2 = \frac{16}{9}z^6 = \frac{16}{9}x^3.$$

Hence we can set $x(t) = t^2$, $y(t) = \frac{4}{3}t^3$, $c(t) = t$. The envelope traces the curve defined by $9y^2 - 16x^3 = 0$ and is tangent to the line $y = 2tx - \frac{2}{3}t^3$ at the point $(t^2, \frac{4}{3}t^3)$ at time $t$.

# 8 Calculating the orthogonal trajectories of a 1-parameter family

Given a 1-parameter family of curves defined by $f(x, y, z) = 0$, we can ask which trajectories are perpendicular to each curve in the family. If $\alpha(t) = (x(t), y(t))$ is an orthogonal trajectory, then we know that for each $t$ there must be $c(t)$ such that $f(x(t), y(t), c(t)) = 0$. In order for $\alpha(t)$ to be heading in a direction perpendicular to the curve defined by $f(x, y, c(t)) = 0$, we must have

$$(x'(t), y'(t)) = \lambda(t)(f_x, f_y)$$

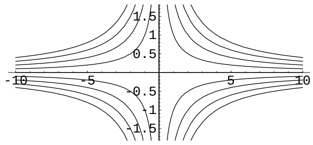
for each $t$ and some $\lambda(t) \neq 0$. So to find $\alpha(t)$ we must solve the system of equations

$$f(x(t), y(t), c(t)) = 0$$

$$(x'(t), y'(t)) \quad = \quad \lambda(t)(f_x, f_y)$$

when $x = x(t)$, $y = y(t)$, $z = c(t)$.

**Example 8.1.** Consider $f(x, y, z) = xy - z$. The curve defined by $f(x, y, c) = 0$ is the hyperbola $xy - c = 0$. Plotting these hyperbolas for $c \in \{-4, -3, \ldots, 3, 4\}$ we obtain




To find the orthogonal trajectories we must solve the system of equations

$$x(t)y(t) - c(t) \quad = \quad 0$$

$$(x'(t), y'(t)) \quad = \quad \lambda(t)(y(t), x(t)).$$

The second equation implies

$$\frac{y'(t)}{x'(t)} = \frac{x(t)}{y(t)},$$

$$y'(t)y(t) = x'(t)x(t),$$

$$(y(t)^2)' = (x(t)^2)',$$

$$y(t)^2 = x(t)^2 + k$$

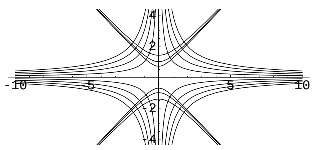
for any fixed number $k$. Having chosen $k$, we can set

$$c(t) = x(t)y(t) = \pm x(t)\sqrt{x(t)^2 + k}.$$

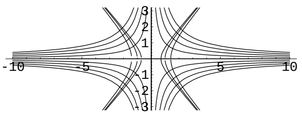
The orthogonal trajectories trace the ellipses

$$y^2 - x^2 = k.$$

The orthogonal trajectories are the 1-parameter family corresponding to $g(x, y, z) = y^2 - x^2 - z$. Superimposing the orthogonal trajectories corresponding to $k \in \{0.5, 1.0, 2.0\}$ we obtain



Superimposing the orthogonal trajectories corresponding to $k \in \{-0.5, -1.0, -2.0\}$ we obtain




# 9 First order differential equations

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given. A first order differential equation is an equation of the form

$$\frac{dy}{dx} = f(x, y).$$

A solution is any function $\phi : (a, b) \to \mathbf{R}$ such that

$$\phi(x) = f(x, \phi(x)).$$

In other words, substituting $y = \phi(x)$ satisfies the equation.

**Example 9.1.** Let $f(x, y) = x$. A solution to the differential equation

$$\frac{dy}{dx} = x$$

is

$$\phi(x) = \frac{x^2}{2} + c, \qquad x \in (-\infty, \infty)$$

for any constant value $c$.

**Example 9.2.** Let $f(x, y) = \frac{x}{y}$. A solution to the differential equation

$$\frac{dy}{dx} = \frac{x}{y}$$

is

$$\phi(x) = \sqrt{x^2 - c}, \qquad x \in (\sqrt{c}, \infty)$$

for any constant value $c \geq 0$. To verify this, observe that

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - c}} = \frac{x}{y}.$$

**Example 9.3.** In Example 7.1 we derived the envelope to the 1-parameter family of curves defined as all solutions to $f(x, y, z) = 0$, where

$$f(x, y, z) = y - 2zx - z^3.$$

The solution turned on finding $\alpha(t) = (x(t), y(t))$ such that

$$(x'(t), y'(t)) = \lambda(t)(-f_y, f_x) = \lambda(t)(-1, -2z(t))$$

and requiring $x = z^2$. This can be reformulated as

$$\frac{y'(t)}{x'(t)} = 2z(t) = 2\sqrt{x(t)}.$$

Setting $x(t) = t$, we are attempting to solve the first order differential equation

$$\frac{dy}{dt} = 2\sqrt{t}.$$

A solution is

$$y = \frac{4}{3}t^{\frac{3}{2}} + c$$

for any constant $c$. The only solution which satisfies $f(x, y, z) = 0$ for the choices $x = t$, $y = \frac{4}{3}t^{\frac{3}{2}} + c$, $z = \sqrt{t}$ is to use $c = 0$.

**Example 9.4.** In Example 8.1 we derived the set of orthogonal trajectories to the 1-parameter family of curves defined as solutions to $f(x, y, z) = 0$, where

$$f(x, y, z) = xy - z.$$

The solution turned on finding $\alpha(t) = (x(t), y(t))$ such that

$$(x'(t), y'(t)) = \lambda(t)(f_x, f_y) = \lambda(t)(y(t), x(t))$$

and requiring $xy = z$. This can be reformulated as

$$\frac{y'(t)}{x'(t)} = \frac{x(t)}{y(t)}.$$

Setting $x(t) = t$, we are attempting to solve the first order differential equation

$$\frac{dy}{dt} = \frac{t}{y}.$$

A solution to this equation, as we found in Example 9.2, is

$$y = \sqrt{t^2 - k}$$

for an arbitrary $k$. The graph of $y$ versus $t$ is orthogonal to the curve defined by $f(x, y, c) = 0$ at the point $(a, \sqrt{a^2 - k})$, where $a$ is any solution to

$$a\sqrt{a^2 - k} = c.$$

**Exercises:**

Section 1.1, Problems 1-10

Section 1.2, Problems 1-4, 6, 8

Section 1.3, Problems 1-5, 7

# 10 Methods for Solving First-Order Differential Equations of the form $M\ dx + N\ dy = 0$

Let $M : \mathbf{R}^2 \to \mathbf{R}$ and $N : \mathbf{R}^2 \to \mathbf{R}$ be given. The notation

$$M\ dx + N\ dy = 0$$

is shorthand for the differential equation

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}.$$

Suppose it is possible to find a function $F : \mathbf{R}^2 \to \mathbf{R}$ such that

$$F_x(x, y) = M(x, y)$$

and

$$F_y(x, y) = N(x, y).$$

Then any trajectory of the form $\alpha(x) = (x, y(x))$ in the surface defined by $F(x, y) = c$ gives rise to a solution to the differential equation

$$M\ dx + N\ dy = 0.$$

To see this, set $f(x) = F(x, y(x))$. Then $f(x) = c$ for all $x$, therefore $f'(x) = 0$ for all $x$. Using the chain rule, we obtain

$$F_x + F_y \frac{dy}{dx} = 0.$$

That is,

$$\frac{dy}{dx} = -\frac{F_y}{F_x} = -\frac{M}{N}.$$

**Example 10.1.** Let $F(x, y) = xy$. The surface $F(x, y) = c$ determines the trajectory $y(x) = \frac{c}{x}$. Using the formula above with $\frac{\partial F}{\partial x} = y$ and $\frac{\partial F}{\partial y} = x$ we must conclude that $y = \frac{c}{x}$ satisfies the differential equation

$$y\ dx + x\ dy = 0.$$

To check that this is correct, note that

$$\frac{dy}{dx} = -\frac{c}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}.$$

Using this idea, we can now solve several types of first-order differential equations of the form $M\ dx + N\ dy = 0$:

1. Exact. Just find $F(x, y)$ so that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. This requires $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Then solve for $y$ in terms of $x$ in the equation $F(x, y) = c$.

2. Separable. These are differential equations which can be put into the form

$$m(x)\ dx + n(y)\ dy = 0.$$

Equations of this form are exact, because $\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x}$. In this case we can use $F(x, y) = \int m(x)\ dx + \int n(y)\ dy$.

3. Homogeneous. These are differential equations which can be put into the form

$$\frac{dy}{dx} = g(y/x).$$

The change of variables $v = \frac{y}{x}$ converts this equation into

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0,$$

which is separable.

4. Linear. These are differential equations which can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Rearranged, this is

$$(P(x)y - Q(x))\ dx + dy = 0.$$

While this equation may not be exact, if we use the integrating factor $\mu = e^{\int P(x)\ dx}$ then the equation

$$\mu(P(x)y - Q(x))\ dx + \mu\ dy = 0$$

is exact. The solution to this equation is the one-parameter family

$$y = \frac{\int \mu Q(x)\ dx + c}{\mu}.$$

5. Bernoulli. These are differential equations which can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

The change of variables $v = y^{1-n}$ converts this to the linear equation

$$\frac{dv}{dx} + (1-n)P(x) = (1-n)Q(x),$$

which can be solved using the methods of 4.

6. Equations in which

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$$

for some $f(x)$. In this case we look for an integrating factor of the form $\mu(x)$. Since we want

$$\mu M\ dx + \mu N\ dy = 0$$

to be exact, we want

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x},$$

$$\mu(x)\frac{\partial M}{\partial y} = \mu'(x)N + \mu(x)\frac{\partial N}{\partial x},$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x),$$

$$\frac{d(\ln \mu(x))}{dx} = f(x),$$

$$\ln \mu(x) = \int f(x)\ dx,$$

$$\mu = e^{\int f(x)\ dx} = e^{\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]\ dx}.$$

Note that 4 is a special case of 6. In this case we have $M = P(x)y - Q(x)$ and $N = 1$, therefore

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = P(x),$$

therefore

$$\mu = e^{\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]\ dx} = e^{\int P(x)\ dx}.$$

Multiplying

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by $\mu$ we obtain

$$\left(e^{\int P(x)\ dx}\right)\frac{dy}{dx} + \left(e^{\int P(x)\ dx}\right)P(x)y = \left(e^{\int P(x)\ dx}\right)Q(x),$$

$$[\left(e^{\int P(x)\ dx}\right)y]' = \left(e^{\int P(x)\ dx}\right)Q(x),$$

$$(\mu y)' = \mu Q,$$

$$\mu y = \int \mu Q\ dx + c,$$

$$y = \frac{\int \mu Q\ dx + c}{\mu}.$$

7. Equations in which

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = f(y)$$

for some $f(y)$. This is just 6 with $x$ and $y$ reversed, and we get the integrating factor

$$\mu = e^{\int f(y)\ dy} = e^{\int \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\ dy}.$$

8. Equations which can be put into the form

$$(ax + by + c)\ dx + (a'x + b'y + c')\ dy = 0.$$

If $\det \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \neq 0$ then the change of variables $u = ax + by + c$ and $v = a'x + b'y + c'$ results in a homogeneous equation in $u$ and $v$. If $\det \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 0$ then the change of variables $u = ax + by + c$ results a separable equation in $x$ and $u$.

**Exercises:**

Section 2.1: Problems: 5, 8, 13, 15, 22-24

Section 2.2: Problems: 5, 11, 12, 14, 17, 25

Section 2.3: Problems: 7, 20, 23, 28, 31, 32, 33, 34, 37, 38, 40

Section 2.4: Problems: 4, 6, 8, 9, 12, 13, 17, 18, 19, 20, 21

Section 3.1: Problems: 5, 6, 9. 10, 11-13, 15, 16, 18

Section 3.2: Problems: 1, 3, 11, 13, 15, 18, 19

Section 3.3: Problems 1, 3, 9, 10, 14, 18, 19, 22, 26, 31, 32

# 11 Methods for solving general order linear differential equations of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = F(x).$$

1. First, find $n$ linearly independent solutions $y = f_1(x)$, ..., $y = f_n(x)$ to the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

The general solution to the homogeneous equation must then be of the form

$$y_c = c_1 f_1(x) + \cdots + c_n f_n(x),$$

where $c_1$ through $c_n$ are arbitrary constants.

2. Second, find one particular solution $y = y_p$ to the original differential equation.

3. The general solution to the original differential equation is then

$$y = y_p + y_c.$$

4. Methods for finding linearly independent solutions to $n^{th}$-order linear homogeneous differential equations:

(4.a) Reduction of order. Assume that $f(x)$ is one solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Another solution to this differential equation is $g(x)f(x)$, where

$$g(x) = \int \frac{1}{f^2 e^{\int \frac{a_1(x)}{a_2(x)}\ dx}}\ dx.$$

More generally, if $f(x)$ is a solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0,$$

and we want to find $g(x)$ so that $g(x)f(x)$ is another solution, then substituting $y = gf$ into the differential equation and noting that

$$(gf)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} g^{(i)} f^{(k-i)}$$

then we can see that $g$ is a solution to the differential equation

$$\sum_{k=0}^{n}\sum_{i=0}^{k} a_k \binom{k}{i} g^{(i)} f^{(k-i)} = 0,$$

or

$$\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} a_k f^{(k-i)}\right) g^{(i)} = 0.$$

Since

$$\sum_{k=0}^{n} a_k f^{(k)} = 0,$$

this leaves us with

$$\sum_{i=1}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} a_k f^{(k-i)}\right) g^{(i)} = 0.$$

Substituting $G = g'$ we obtain

$$\sum_{i=1}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} a_k f^{(k-i)}\right) G^{(i-1)} = 0.$$

Therefore $G$ satisfies the $(n-1)^{st}$-order linear differential equation

$$b_{n-1}(x)y^{(n-1)} + \cdots + b_0(x)y = 0,$$

where

$$b_i(x) = \sum_{k=i+1}^{n}\binom{k}{i+1} a_k f^{(k-i-1)}$$

for each $i$. Solve this for $G$, then set

$$g = \int G(x)\ dx.$$

(4.b) Equations with constant coefficients. If we guess $y = e^{rx}$ as a solution to

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y^{(0)} = 0,$$

then after substitution we obtain

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} = 0.$$

Dividing by $e^{rx}$, we find that

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0.$$

This can be factored into the form

$$a_n (r - r_1)(r - r_2) \cdots (r - r_n) = 0.$$

Therefore the solutions are

$$r = r_1, r_2, \ldots, r_n.$$

Since the coefficients $a_i$ are real, any complex roots must occur in complex-conjugate pairs. For example, suppose we find are dealing with a $11^{th}$-order linear homogeneous differential equation with constant coefficients, and we find that the solutions to $r$ are

$$r = 1, 1, 1, 2, 2, 3 + 4i, 3 + 4i, 3 - 4i, 3 - 4i, 5i, -5i.$$

Then 11 linearly independent solutions are

$$e^x, xe^x, x^2 e^x, e^{2x}, xe^{2x}, e^{3x}\cos(4x), xe^{3x}\cos(4x), e^{3x}\sin(4x), xe^{3x}\sin(4x), \cos(5x), \sin(5x).$$

5. Methods of finding one particular solution to

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = F(x).$$

(5.a) Undetermined coefficients. The idea is to guess at the form of the solution, then plug in and determine the unknown coefficients. Example:

$$y'' - 4y + 4y = xe^{2x}.$$

First find the general solution to the homogeneous equation:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

Then form the UC set corresponding to $F(x)$:

$$xe^{2x} \to \{e^{2x}, xe^{2x}\}.$$

Multiply this set by the lowest non-negative power of $x$ needed so that none of the resulting functions is a solution to the homogeneous equation:

$$x^2\{e^{2x}, xe^{2x}\} = \{x^2e^{2x}, x^3e^{2x}\}.$$

Combine the functions in the UC set to create a particular solution $y_p$ with undetermined coefficients:

$$y_p = Ax^2e^{2x} + Bx^3e^{2x}.$$

Now determine the coefficients by requiring $y_p$ to satisfy the differential equation:

$$(Ax^2e^{2x} + Bx^3e^{2x})'' - 4(Ax^2e^{2x} + Bx^3e^{2x})' + 4(Ax^2e^{2x} + Bx^3e^{2x}) = xe^{2x}.$$

After taking all the derivatives and simplifying the left-hand side of this equation, we obtain

$$2Ae^{2x} + 6Bxe^{2x} = xe^{2x}.$$

Equating coefficients, we have

$$2A = 0, 6B = 1.$$

Now we have

$$y_p = \frac{1}{6}x^3e^{2x}.$$

(5.b) Divide and Conquer. Suppose we need to find a particular solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = F_1(x) + F_2(x) + \cdots + F_k(x).$$

Let $y_i$ be a particular solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = F_i(x)$$

for $1 \leq i \leq k$. Then

$$y_p = y_1 + y_2 + \cdots + y_k$$

is a particular solution to the original differential equation. For example, suppose we want a particular solution to

$$y'' - 4y' + 4y = xe^{2x} + x^2.$$

We know that

$$y_1 = \frac{1}{6}x^3e^{2x}$$

is a solution to

$$y'' - 4y' + 4y = xe^{2x},$$

and it is easy to verify that

$$y_2 = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$$

is a solution to

$$y'' - 4y' + 4y = x^2,$$

therefore

$$y_p = y_1 + y_2 = \frac{1}{6}x^3e^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$$

is a solution to

$$y'' - 4y' + 4y = xe^{2x} + x^2.$$

(5.c) Variation of parameters. We will illustrate the method for the second-order linear equation

$$y'' + a_1(x)y' + a_0(x)y = F(x).$$

First find two linearly independent solutions $f_1(x)$ and $f_2(x)$ to the homogeneous equation

$$y'' + a_1(x)y' + a_0(x)y = 0.$$

Now we attempt to find $g_1(x)$ and $g_2(x)$ so that

$$y_p = g_1f_1 + g_2f_2$$

is a particular solution to the original differential equation. Plugging in $y_p$ we obtain

$$(g_1f_1 + g_2f_2)'' + a_1(g_1f_1 + g_2f_2)' + a_0(g_1f_1 + g_2f_2) = F(x).$$

Therefore

$$\begin{gathered}(g_1''f_1 + 2g_1'f_1' + g_1f_1'' + g_2''f_2 + 2g_2'f_2' + g_2f_2'')+\\ a_1(g_1'f_1 + g_1f_1' + g_2'f_2 + g_2f_2')+\\ a_0(g_1f_1 + g_2f_2) = F(x).\end{gathered}$$

However, we know that

$$f_1'' + a_1f_1' + a_0f_1 = f_2'' + a_1f_2' + a_0f_2 = 0,$$

therefore we are left with

$$(g_1''f_1 + 2g_1'f_1' + g_2''f_2 + 2g_2'f_2') + a_1(g_1'f_1 + g_2'f_2) = F(x).$$

To simplify this, we will impose the condition that

$$g_1'f_1 + g_2'f_2 = 0.$$

This leaves us with

$$g_1''f_1 + 2g_1'f_1' + g_2''f_2 + 2g_2'f_2' = F(x).$$

Note however that we have

$$(g_1'f_1 + g_2'f_2)' = 0,$$

$$g_1''f_1 + g_1'f_1' + g_2''f_2 + g_2'f_2' = 0,$$

therefore

$$g_1'f_1' + g_2'f_2' = F(x).$$

In summary, $g_1$ and $g_2$ must satisfy two equations:

$$\begin{aligned}g_1'f_1 + g_2'f_2 &= 0\\ g_1'f_1' + g_2'f_2' &= F.\end{aligned}$$

Solving for $g_1'$ and $g_2'$ we obtain

$$g_1' = \frac{-f_2F}{f_1f_2' - f_1'f_2} = \frac{-f_2F}{W(f_1,f_2)}, \qquad g_2' = \frac{f_1F}{f_1f_2' - f_1'f_2} = \frac{f_1F}{W(f_1,f_2)}.$$

Therefore

$$g_1 = \int \frac{-f_2F}{W(f_1,f_2)}\, dx, \qquad g_2 = \int \frac{f_1F}{W(f_1,f_2)}\, dx,$$

and finally

$$y_p = f_1 \int \frac{-f_2 F}{W(f_1, f_2)}\, dx + f_2 \int \frac{f_1 F}{W(f_1, f_2)}\, dx.$$

6. The Cauchy-Euler equation. This is any $n^{th}$-order linear differential equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y^{(0)} = F(x).$$

The associated homogeneous linear differential equation is

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y^{(0)} = 0.$$

If we make the change of variables $x = e^t$ and guess $y = e^{rt}$, then we obtain

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{re^{rt}}{e^t} = re^{(r-1)t},$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d(\frac{dy}{dx})}{dt}}{\frac{dx}{dt}} = \frac{r(r-1)e^{(r-1)t}}{e^t} = r(r-1)e^{(r-2)t},$$

and finally

$$\frac{d^n t}{dx^n} = r(r-1)\cdots(r-n+1)e^{(r-n)t}.$$

Therefore for all $k$ we have

$$x^k y^{(k)} = r(r-1)\cdots(r-k+1)e^{rt} = r(r-1)\cdots(r-k+1)y.$$

Therefore the homogeneous equation becomes

$$a_n[r(r-1)\cdots(r-n+1)]y + a_{n-1}[r(r-1)\cdots(r-n)]y + \cdots + a_1 ry + a_0 y = 0,$$

and letting $r = r_1, r_2, \ldots, r_n$ be the solutions to

$$a_n[r(r-1)\cdots(r-n+1)] + a_{n-1}[r(r-1)\cdots(r-n)] + \cdots + a_1 r + a_0 = 0,$$

we can find $n$ linearly independent solutions $y = f_1(t), y = f_2(t), \ldots, y = f_n(t)$ to the homogeneous equation. To solve the original linear differential equation, we can use either undetermined coefficients or variation of parameters.

Example:

$$x^2y'' + xy' + 4y = 2x \ln x.$$

The homogeneous equation is

$$x^2y'' + xy' + 4y = 0.$$

Setting $x = e^t$ and guessing $y = e^{rt}$ for the homogeneous solution we obtain

$$r(r-1) + r + 4 = 0,$$

$$r^2 + 4 = 0,$$

$$r = 0 + 2i, 0 - 2i.$$

Therefore the general solution to the homogenous equation is

$$y_c = c_1 \cos(2t) + c_2 \sin(2t).$$

Since $F(e^t) = 2te^t$, we can use variation of parameters with $W(\cos(2t), \sin(2t)) = 2$ to obtain

$$y_p = \cos(2t) \int \frac{-\sin(2t) \cdot 2te^t}{2}\, dt + \sin(2t) \int \frac{\cos(2t) \cdot 2te^t}{2}\, dt,$$

which looks hard to simplify. Another approach is to use undetermined coefficients: the UC set of $2te^t$ is $\{te^t, e^t\}$, which does not need to be modified. If we guess

$$y_p = Ae^t + Bte^t = Ax + Bx \ln x,$$

then

$$x^2y_p'' + xy_p' + 4y_p = 2x \ln x$$

becomes

$$x^2 \frac{B}{x} + x(A + B \ln x + B) + 4(Ax + Bx \ln x) = 2x \ln x.$$

Equating coefficients of $x$ and $x \ln x$, we find

$$5A + 2B = 0, 5B = 2,$$

hence

$$A = -\frac{4}{25}, B = \frac{2}{5},$$

$$y_p = -\frac{4}{25}x + \frac{2}{5}x \ln x.$$

In this case the general solution to the original differential equation is

$$y = y_p + y_c = -\frac{4}{25}x + \frac{2}{5}x \ln x + c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x).$$

**Exercises:**

Section 4.1, page 122: Problems 1,2, 3, 9, 10, 11

Section 4.1, page 132: Problems 1, 5, 9, 11

Section 4.2: Problems 1, 11, 26, 32, 37, 51, 55, 57, 59, 61

Section 4.3: Problems 1, 11, 21, 31, 41, 51, 61

Section 4.4: Problems 1, 3, 13, 17, 19, 25

Section 4.5: Problems 1, 6, 11, 16, 21, 26, 31

Section 5.2: Problems 1, 3, 5, 7, 8, 9

Section 5.3: 1, 3, 5, 7, 9, 11, 13

Section 5.4: 1, 3, 5, 7, 9

# 12 Systems of two linear differential equations with independent variables $x$ and $y$

1. Review of second order linear differential equations:

$$(E) \qquad a_2(t)x'' + a_1(t)x' + a_0(t) = F(t).$$

The homogeneous equation is

$$(H) \qquad a_2(t)x'' + a_1(t)x' + a_0(t) = 0.$$

Two linearly independent solutions $f_1$, $f_2$ to $(H)$ are guaranteed, and the general solution to $(H)$ is

$$x_c = c_1 f_1 + c_2 f_2.$$

If $x_p$ is any particular solution to $(E)$, then the general solution to $(E)$ is $x = x_c + x_p$.

2. Transition to systems of two first order linear differential equations:

The second order linear differential equation $(E)$ can be converted to a system of two first order linear differential equations:

$$(E) \qquad \begin{cases} y & = x' \\ \\ a_2(t)y' + a_1(t)y + a_0(t)x & = F(t). \end{cases}$$

We will rearrange this to

$$(E) \qquad \begin{cases} x' & = y \\ \\ y' & = -a_0(t)x - a_1(t)y + F(t). \end{cases}$$

In matrix form:

$$(E) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}.$$

3. General systems of two first order linear differential equations:

$$(E) \qquad \begin{cases} x' & = a_{11}(t)x + a_{12}(t)y + F_1(t) \\ \\ y' & = a_{21}(t)x + a_{22}(t)y + F_2(t). \end{cases}$$

In matrix form:

$$(E) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}.$$

More simply:

$$(E) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = A(t) \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

4. Theory of systems of two first order linear differential equations.

(4.1) Theorem 7.1: The boundary value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A(t) \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

has a unique solution, given the appropriate hypotheses on the coefficient functions and forcing functions.

(4.2) Notation: $(E)$ and $(H)$ are the original differential equation and the associated homogeneous differential equation:

$$(E) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = A(t) \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},$$

$$(H) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = A(t) \begin{bmatrix} x \\ y \end{bmatrix}.$$

(4.3) Theorem 7.3: The homogeneous equation $(H)$ has two linearly independent solutions:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_2 \\ g_2 \end{bmatrix}.$$

The general solution to $(H)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} + c_2 \begin{bmatrix} f_2 \\ g_2 \end{bmatrix}.$$

(4.4) The Wronskian of $\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}$ and $\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}$:

$$W(t) = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}.$$

(4.5) Theorem 7.4: $\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}$ and $\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}$ are linearly independent if and only if $W(t) \neq 0$ at some point $t$.

(4.6) Theorem 7.6: Let

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

be any particular solution to $(E)$, and let

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix}$$

be the general solution to $(H)$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \end{bmatrix} + \begin{bmatrix} x_c \\ y_c \end{bmatrix}$$

is the general solution to $(E)$.

5. Method for solving $(H)$ if the matrix $A(t)$ has constant coefficients:

$$(H) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

(5.1) Guess

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Substituting into $(H)$, obtain

$$\lambda e^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix} = A e^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Therefore

$$\lambda \begin{bmatrix} P \\ Q \end{bmatrix} = A \begin{bmatrix} P \\ Q \end{bmatrix}.$$

In other words, $\begin{bmatrix} P \\ Q \end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. From matrix theory, to find $\lambda$ we need to solve the determinant equation

$$\left| A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0.$$

Having found $\lambda$, we can then solve for $P$ and $Q$.

(5.2) Two real roots $\lambda_1$ and $\lambda_2$:

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}.$$

(5.3) Complex conjugate roots $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$: multiply out

$$e^{(a+bi)t} \begin{bmatrix} P_1 + iP_2 \\ Q_1 + iQ_2 \end{bmatrix}$$

using

$$e^{a+bi} = e^a \cos at + ie^a \sin bt$$

and separate into real and imaginary parts. These are two linearly independent solutions.

(5.4) Repeated real roots $\lambda$, $\lambda$: Suppose a solution of

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}$$

is obtained. Look for a second solution of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} P' \\ Q' \end{bmatrix} + te^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

It must satisfy

$$(H) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

Making the substitution, it must satisfy

$$\lambda e^{\lambda t} \begin{bmatrix} P' \\ Q' \end{bmatrix} + (e^{\lambda t} + \lambda te^{\lambda t}) \begin{bmatrix} P \\ Q \end{bmatrix} = Ae^{\lambda t} \begin{bmatrix} P' \\ Q' \end{bmatrix} + Ate^{\lambda t} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Dividing by $e^{\lambda t}$, we must have

$$\lambda \begin{bmatrix} P' \\ Q' \end{bmatrix} + (1+\lambda t) \begin{bmatrix} P \\ Q \end{bmatrix} = A \begin{bmatrix} P' \\ Q' \end{bmatrix} + At \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Since

$$At \begin{bmatrix} P \\ Q \end{bmatrix} = \lambda t \begin{bmatrix} P \\ Q \end{bmatrix},$$

we can subtract this quantity from both sides to obtain

$$\lambda \begin{bmatrix} P' \\ Q' \end{bmatrix} + \begin{bmatrix} P \\ Q \end{bmatrix} = A \begin{bmatrix} P' \\ Q' \end{bmatrix}.$$

Therefore we must solve

$$\left( A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} P' \\ Q' \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix}.$$