Week 15 Lectures Math 223 Fall 2009

Stoke's Theorem:

$$\int_C F \cdot dr = \int\int_S \text{curl } F \cdot dS.$$

Proof: Let $r(u,v) = (x,y,z)$ be paramaterization of S. Let C be closed path in S. We'll assume that this comes from a path C' in the uv-plane: $(u(t), v(t))$. So path is $(x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t)))$. Let D be the region enclosed by C on S and let D' be the region enclosed by C' in the uv plane. If $F = (P,Q,R)$, then

$$\int_C F \cdot dr = \int_C P\ dx + Q\ dy + R\ dz =$$

$$\int_a^b P(x_u u' + x_v v') + Q(y_u u' + y_v v') + R(z_u u' + z_v v')\ dt =$$

$$\int_a^b (Px_u + Qy_u + Rz_u)u' + (Px_v + Qy_v + Rz_v)v'\ dt =$$

$$\int_{C'} (Px_u + Qy_u + Rz_u)\ du + (Px_v + Qy_v + Rz_v)\ dv =$$

$$\int\int_{\partial D'} (Px_v + Qy_v + Rz_v)_u - (Px_u + Qy_u + Rz_u)_v\ dA$$

by Green's Theorem. Now

$$(Px_v)_u = P_u x_v + P x_{vu} = (P_x x_u + P_y y_u + P_z z_u)x_v + P x_{vu} = P_x x_u x_v + P_y y_u x_v + P_z z_u x_v + P x_{vu}$$

$$(Qy_v)_u = Q_u y_v + Q y_{vu} = (Q_x x_u + Q_y y_u + Q_z z_u)y_v + Q y_{vu} = Q_x x_u y_v + Q_y y_u y_v + Q_z z_u y_v + Q y_{vu}$$

$$(Rz_v)_u = R_u z_v + R z_{vu} = (R_x x_u + R_y y_u + R_z z_u)z_v + R z_{vu} = R_x x_u z_v + R_y y_u z_v + R_z z_u z_v + R z_{vu}$$

$$(Px_u)_v = P_v x_u + P x_{uv} = (P_x x_v + P_y y_v + P_z z_v)x_u + P x_{uv} = P_x x_v x_u + P_y y_v x_u + P_z z_v x_u + P x_{uv}$$

$$(Qy_u)_v = Q_v y_u + Q y_{uv} = (Q_x x_v + Q_y y_v + Q_z z_v)y_u + Q y_{uv} = Q_x x_v y_u + Q_y y_v y_u + Q_z z_v y_u + Q y_{uv}$$

$$(Rz_u)_v = R_v z_u + R z_{uv} = (R_x x_v + R_y y_v + R_z z_v)z_u + R z_{uv} = R_x x_v z_u + R_y y_v z_u + R_z z_v z_u + R z_{uv}$$

hence

$$(Px_v + Qy_v + Rz_v)_u - (Px_u + Qy_u + Rz_u)_v =$$

$$P_y(y_u x_v - y_v x_u) + P_z(z_u x_v - z_v x_u) +$$
$$Q_x(x_u y_v - x_v y_u) + Q_z(z_u y_v - z_v y_u) +$$
$$R_x(x_u z_v - x_v z_u)) + R_y(y_u z_v - y_v z_u) =$$
$$(R_y - Q_z)(y_u z_v - y_v z_u) - (P_z - R_x)(x_u z_v - x_v z_u) + (Q_x - P_y)(x_u y_v - x_v y_u) =$$
$$(R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot (y_u z_v - y_v z_u, -(x_u z_v - x_v z_y), x_u y_v - x_v y_u) =$$
$$\text{curl } F \cdot (r_u \times r_v).$$

Therefore

$$\int\int_{\partial D'} (P x_v + Q y_v + R z_v)_u - (P x_u + Q y_u + R z_u)_v \; dA = \int\int_{\partial D'} \text{curl } F \cdot (r_u \times r_v) \; dA$$

$$= \int\int_S \text{curl } F \cdot dS.$$