Week 14 Lectures Math 223 Spring 2008

Section 17.7: Surface Integrals

Surface Integral: Let S be surface in xyz coordinate system, which is the image of the region Ω in the uv plane with respect to the parameterization $r(u,v) = (x(u,v), y(u,v), z(u,v))$. Take your surface, chop it up into little surfaces, pick a representative point in the little surface, multiply the integrand at this point by the area of the little surface, add together. In the limit, obtain

$$\int\int_S H(x,y,z)\ d\sigma = \int\int_\Omega H(u,v)||r_u \times r_v||\ dudv$$

where $r_u = (x_u, y_y, v_u)$ and $r_v = (x_v, y_v, z_v)$. Dividing a surface integral by the surface area gives the average value of the integrand over the surface. Given a mass density function as the integrand, we can compute the mass of the surface. We can also compute the centroid of a uniform density surface and the center of mass of a variable density surface using surface integrals.

Flux of a vector field across an oriented surface: Center of mass coordinate $\overline{x}$ can be viewed as average value of x-coordinates in the surface, weighted by area distribution. Another average value we can compute is the component of the a vector field $F(x,y,z) = (P,Q,R)$ in the direction of the unit normal to the surface $n(x,y,z)$. The component is $F(x,y,z) \cdot n(x,y,z)$, and the average value is

$$\frac{\int\int F(x,y,z) \cdot n(x,y,z)||r_u \times r_v||}{\int\int ||r_u \times r_v||.}$$

The numerator is called the flux of F across the surface. Since the unit normal vector can be computed as

$$n(x,y,z) = \frac{r_u \times r_v}{||r_u \times r_v||},$$

we have

$$\text{flux} = \int\int F(x,y,z) \cdot (r_u \times r_v) = \int\int \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$

Using $u = x$ and $v = y$ and the surface $z = f(x, y)$ we get

$$\text{flux} = \int\int \begin{vmatrix} P & Q & R \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}.$$

See also page 1071.

Geometric interpretation: If $F = (P, Q, R)$ represents a velocity vector of a current in units of feet per second, then $F \cdot n$ denotes speed of current in the direction of the normal vector across the surface and has units of feet per second. The surface area has units of square feet. Flux is average value of $F \cdot n$ times the area of the surface, so it has units of ft^3 sec. It can be interpreted as rate at which the current is crossing the surface in the direction of the unit vector.

Section 17.9: The Divergence Theorem

Divergence of the vector field $F = (F_1, F_2, F_3)$ **at the point** (x, y, z)**:** Let S denote the surface of a rectangular box with vertex at (x, y, z) and dimensions $\Delta x \times \Delta y \times \Delta z$ extending in the positive directions away from (x, y, z). The approximate divergence of F at (x, y, z) is the average flux of F across all 6 faces of the box per unit volume:

$$\text{average div } F = \frac{\int\int\int_S F \cdot n \; d\sigma}{\Delta x \Delta y \Delta z}.$$

The exact divergence at (x, y, z) is the limit of these calculations as the box shrinks to the point (x, y, z).

Divergence Calculation: Let the box have vertex at (x, y, z) and extending by Δx, Δy, and Δz units in the coordinate directions. Use the value of the vector field F at the 3 corners of the box connected to (x, y, z) to approximate the strength of the vector field along the 6 faces. The flux calculation yields

$$\int\int\int_S F \cdot n \; d\sigma \approx (F_1(x + \Delta x, y, z) - F_1(x, y, z))\Delta y \Delta z +$$

$$(F_2(x, y + \Delta y, z) - F_2(x, y, z))\Delta x \Delta z + (F_3(x, y, z + \Delta x) - F_x(x, y, z))\Delta x \Delta y.$$

The average divergence is therefore

$$\frac{\int\int\int_S F \cdot n \; d\sigma}{\Delta x \Delta y \Delta z} \approx \frac{F_1(x + \Delta x, y, z) - F_1(x, y, z)}{\Delta x} + \frac{F_2(x, y + \Delta y, z) - F_2(x, y, z)}{\Delta y} +$$

$$\frac{F_3(x,y,z+\Delta x)-F_x(x,y,z)}{\Delta x}.$$

Taking the limit as $\Delta x, \Delta y, \Delta z \to 0$, we obtain

$$\text{div } F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}.$$

We can now estimate flux across a small box as

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\right)\Delta x \Delta y \Delta z.$$

Divergence Theorem: Flux across the surface of a solid region Ω can be expressed as the triple integral of the divergence. In symbols,

$$\int\int_{\text{surface of }\Omega} F \cdot n \; d\sigma = \int\int\int_{\Omega}\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\right) \; dxdydz.$$

Proof: Push a bunch of boxes together with corners at (x_i, y_i, z_i) and dimensions $\Delta x \times \Delta y \times \Delta z$ to approximate the solid region Ω and add up the total flux across the surfaces of the boxes. The contribution to flux across faces which oppose each other cancel out, and the net flux is across the outer surfaces of the boxes. In this way we get an approximation to the flux across the surface of Ω:

$$\int\int_{\text{surface of }\Omega} F \cdot n \; d\sigma \approx$$

$$\sum_i \left(\frac{\partial F}{\partial x}(x_i, y_i, z_i) + \frac{\partial F}{\partial y}(x_i, y_i, z_i) + \frac{\partial F}{\partial z}(x_i, y_i, z_i)\right)\Delta x \Delta y \Delta z.$$

In the limit we get the flux across the surface of Ω, which can be expressed as the triple integral

$$\int\int\int_{\Omega}\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\right) \; dxdydz.$$

Note that this can be used to compute surface area via a triple integral. Just use $F = n$. Consider the sphere of radius r regarded as a level surface of $f(x,y,z) = x^2 + y^2 + z^2$. The gradient is $(2x, 2y, 2z)$, hence $n(x,y,z) = \frac{1}{r}(x,y,z)$. The divergence of n is $\frac{3}{4}$. So the surface area of the sphere is $\int\int\int_{\Omega} \frac{3}{r} \; dxdydz$, where Ω is the region inside the unit sphere. This produces $\frac{3}{r}$ times the volume of the sphere of radius r. Since the volume of the unit sphere is $\frac{4}{3}\pi r^3$, the surface area of the sphere is $4\pi r^2$.