Week 13 Lectures

Sections 13.6 and 13.7

Section 13.6: Parametric Surfaces and their Areas

Idea: use two variables to describe surface in 3-dimensions.

Example: Graph of $z = f(x, y)$ can be viewed as all $(x, y, f(x, y))$.

Example: Surface of cone can be represented as $(r\cos\theta, r\sin\theta, r)$

Example: Surface of sphere can be represented as $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)$

Example: Take a circle of radius r centered at $(R, 0, 0)$ parallel to the xz-plane. The typical point point on the torus is found by first rotating the point at $(R+r, 0, 0)$ through an angle of θ along the circle, then along an angle of ϕ about the z-axis. θ action takes us to the point $(R, 0, 0)+(r\cos\theta, 0, r\sin\theta) = (R+r\cos\theta, 0, r\sin\theta)$. Now rotate about the z-axis through angle ϕ and keep at the same z-level: ignoring the z-coordinate, we are describing a circle in the xy-plane with radius $R+r\cos\theta$, and we are brought to $x = (R+r\cos\theta)\cos\phi$, $y = (R + r\cos\theta)\sin\phi$. Hence the typical point on the torus is

$$((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta), \qquad 0 \le \theta \le 2\pi, 0 \le \phi \le 2\pi.$$

Tangent plane equation: the typical parametric surface is

$$r(u, v) = (x(u, v), y(u, v), z(u, v).$$

We wish to find the equation of plane tangent to the surface at the point $r(u_0, v_0)$. All we need to do is find two tangent vectors and take their cross product to find the normal to the tangent plane. Two points on the surface are $r(u_0, v_0)$ and $r(u_0 + h, v_0)$. The displacement is approximately $(x_u h, y_u h, z_u h) = hr_u$. Two other points on the surface are $r(u_0, v_0)$ and $r(u_0, v_0 + h)$. The displacement is approximately $(x_v h, y_v h, z_v h) = hr_v$. So two vectors in the tangent plane are approximately r_u and r_v, evaluated at (u_0, v_0). The normal is $r_u \times r_v$. So the tangent plane equation is

$$(r_u \times r_v)(x - x_0, y - y_0, z - z_0) = 0.$$

Surface area calculation: Going back to these displacements we form a parallelogram with area $||hr_u \times hr_v|| = ||r_u \times r_v||h^2$. Adding those up we get approximate surface area. But we are computing a double integral in the uv plane, so we gev

$$A(S) = \int\int_D ||r_u \times r_v||\ dA.$$

Surface area of the torus:

$$\int_0^{2\pi}\int_0^{2\pi} r^2(R + r\cos\theta)^2\ d\theta\ d\phi = 2\pi^2 r^2(r^2 + 2R^2).$$

Surface area of the graph of a function: see Formula 9, page 772.

Section 13.7: Surface Integrals

Mass of a surface parameterized by $r(u, v)$, given mass density function $f(x, y, z)$: multiply each little parallelogram by mass density. Get $\sum f(r(u, v))||r_u \times r_v||h^2$, exact answer

$$\int\int_D f(r(u, v)||r_u \times r_v||\ dA$$

where D describes how u and v vary.

Parameterizing the surface consisting of the cylinder $x^2 + z^2 = 0$ bounded by the planes $y = 0$ and $x + y = 2$: $x = r\cos\theta$, $z = r\sin\theta$, $y = t(2 - r\cos\theta)$, $0 \le \theta \le 2\pi$, $0 \le t \le 1$.

Notation: the surface area $\int\int_S f\ dS$.

Speed in a given direction: If your velocity vector is (a, b) then you are traveling in the x direction at a feet/second, in the y direction at b feet/second, and in the u direction at $(a, b) \cdot u$ feet per second, where u is a unit vector. Now if water if flowing across a flat surface S with area A square feet with velocity (a, b) in the direction u, then the rate of water passing across the surface is $A(a, b) \cdot u$ feet cubed per second. Call this flux. If F is a vector field representing velocity of water at each position in space and S is a surface, we can calculate the flux of water across the surface by chopping up the surface into little parallelograms, treating direction of the water as normal to the surface, and using this formula. Flux across a parallelogram is

$$F \cdot \frac{r_u \times r_v}{||r_u \times r_v||}||r_u \times r_v||h^2.$$

Adding, we arrive at the surface integral

$$\int_S F \cdot n \; dS.$$

The flux is also the sum of $F \cdot (r_u \times r_v)$ contributions, so an alternate notation is

$$\int_S F \cdot dS.$$