Week 12 Lectures

Sections 13.4 and 13.5

Section 13.4: Green's Theorem

Green's Theorem: Let C be positively oriented simple closed curve which encloses region D. Then

$$\int_C P\ dx + Q\ dy = \int\int_D Q_x - P_y\ dA.$$

Proof: Let $F(x,y) = (P,Q)$. This is equivalent to proving

$$\int_C F \cdot dr = \int\int_D Q_x - P_y\ dA.$$

First consider C the rectangle with lower corner at (a,b) and extending right and up Δ units. Decompose C into $C_1+C_2+C_3+C_4$. Then we have

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr + \int_{C_4} F \cdot dr.$$

Approximate the functions $P(x,y)$ and $Q(x,y)$ by

$$P(x,y) \approx P(a,b) + P_x(a,b)(x-a) + P_y(a,b)(y-b)$$

and

$$Q(x,y) \approx Q(a,b) + Q_x(a,b)(x-a) + Q_y(a,b)(y-b).$$

Then we have

$$F(x,y) \approx F(a,b) + (x-a)(P_x(a,b), Q_x(a,b)) + (y-b)(P_y(a,b), Q_y(a,b)).$$

Let C_1 be represented by $r(t) = (a+t, b)$ where $0 \leq t \leq \Delta$. Then

$$F(r(t)) \cdot r'(t) = (F(a,b) + t(P_x(a,b), Q_x(a,b))) \cdot (1,0) = P(a,b) + tP_x(a,b).$$

Let C_2 be represented by $r(t) = (a+\Delta, b+t)$ where $0 \leq t \leq \Delta$. Then

$$F(r(t)) \cdot r'(t) = (F(a,b) + \Delta(P_x(a,b), Q_x(a,b)) + t(P_y(a,b), Q_y(a,b))) \cdot (0,1) =$$

$$Q(a,b) + \Delta Q_x(a,b) + tQ_y(a,b).$$

Let C_3 be represented by $r(t) = (a + \Delta - t, b + \Delta)$ where $0 \le t \le \Delta$. Then

$$F(r(t)) \cdot r'(t) = (F(a,b) + (\Delta - t)(P_x(a,b), Q_x(a,b)) + \Delta(P_y(a,b), Q_y(a,b))) \cdot (-1, 0) =$$

$$-P(a,b) + (-\Delta + t)P_x(a,b) - \Delta P_y(a,b).$$

Let C_4 be represented by $r(t) = (a, b + \Delta - t)$ where $0 \le t \le \Delta$. Then

$$F(r(t)) \cdot r'(t) = (F(a,b) + (\Delta - t)(P_y(a,b), Q_y(a,b))) \cdot (0, -1) =$$

$$-Q(a,b) + (-\Delta + t)Q_y(a,b).$$

Therefore

$$\int_{C_1} F \cdot dr + \int_{C_2} F \cdot + \int_{C_3} F \cdot dr + \int_{C_4} F \cdot dr =$$

$$\int_0^\Delta (2t - \Delta)(P_x + Q_y) + (Q_x - P_y)\Delta \; dt = (Q_x - P_y)\Delta^2.$$

Now let D be covered by squares of dimension $\Delta \times \Delta$. When we add together the work done around all the little rectangles, opposing sides cancel, leaving work done around the boundary of D, namely C. On the other hand, we are adding together $(Q_x - P_y)\Delta^2$, so computing the mass of D with mass density function $Q_x - P_y$. Hence the theorem.//

Note that by choosing P and Q such that $Q_x - P_y = 1$ we have a way to compute area.

Section 13.5: Curl and Divergence.

Let $F = (P, Q, R)$ be a vector field. The curl of F is a new vector field:

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Note that curl $F \cdot k = Q_x - P_y$, an expression which appears in Green's Theorem. So Green's Theorem can be re-written

$$\int_C F \cdot dr = \int \int_D \text{curl } F \cdot k \; dA.$$

Translation: the work done by F in the direction of C is equal to the double integral of the normal component of the curl across the region enclosed by C. This suggests that Green's Theorem is a special case of a 3-dimensional theorem.

The curl of ∇f is 0. Reason: mixed-partial derivatives are equal. If a vector field is defined on all of $\mathbb{R}^3$ and its curl is zero everywhere, then $F = \nabla f$ for some scalar field f, hence is conservative. Proved later using Stoke's Theorem, section 13.8.

Method to decide if F is conservative: Want to show that $F = \nabla f$. If so, the curl of F is 0. So if not, F is not conservative. Note also that if the curl IS zero, then F is conservative (but it takes Stokes's Theorem in Section 13.8 to prove this).

Divergence of F is $F_x + F_y + F_z$. So the divergence of a vector field is a scalar field.

Now consider $F(x, y) = (P(x, y), Q(x, y))$. By Green's Theorem,

$$\int\int_D \text{div } F \, dA = \int\int_D Q_y + P_x \, dA = \int_C -Q \, dx + P \, dy =$$

$$\int_C (P, Q) \cdot (-y'(t), x'(t)) \, dt = \int_C F \cdot n \, ds$$

where

$$n(t) = \frac{(-y', x'(t))}{|r'(t)|} = \text{ unit normal to } C.$$

Translation: The line integral of the normal component of F about C is equal to the double integral of the divergence of F across the region.

The divergence of the curl F is zero. Reason: mixed partial derivatives are equal. This yields a method to test if a vector field is the curl of another: its divergence must be zero.