Week 11 Lectures Math 223

Section 16.6: Triple Integrals

Example: compute mass and center of mass. Gives rise to Riemann sum over rectangular solids.

Section 16.7: Reduction to Repeated Integrals

Assume function being integrated is $f(x, y, z)$, over region $a_1 \leq x \leq a_2$, $\phi_1(x) \leq y \leq \phi_2(x)$, $\psi_1(x, y) \leq z \leq \psi_2(x, y)$. The Riemann sum is

$$\sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\Delta x \Delta y \Delta z.$$

The x_1^* contribution is

$$\sum_{j,k} f(x_1^*, y_j^*, z_k^*)\Delta y \Delta z \times \Delta x,$$

the left-hand factor of which is the Riemann sum for the type I double integral with coordinate axes y, z, and $w = f(x_1^*, y, z)$. Hence x_1^* contribution is approximately

$$\int_{y=\phi_1(x_1^*)}^{y=\phi_2(x_1^*)} \left[\int_{z=\psi_1(x_1^*,y)}^{z=\psi_1(x_1^*,y)} f(x_1^*, y, z)\ dz \right] dy \times \Delta x = F(x_1^*)\Delta x$$

where

$$F(x) = \int_{y=\phi_1(x)}^{y=\phi_2(x)} \left[\int_{z=\psi_1(x,y)}^{z=\psi_1(x,y)} f(x, y, z)\ dz \right] dy.$$

Similarly, the x_2^* contribution is

$$\sum_{j,k} f(x_2^*, y_j^*, z_k^*)\Delta y \Delta z \times \Delta x,$$

the left-hand factor of which is the Riemann sum for the type I double integral with coordinate axes y, z, and $w = f(x_2^*, y, z)$. Hence the x_2^* contribution is approximately

$$\int_{y=\phi_1(x_2^*)}^{y=\phi_2(x_2^*)} \left[\int_{z=\psi_1(x_2^*,y)}^{z=\psi_1(x_2^*,y)} f(x_2^*, y, z)\ dz \right] dy \times \Delta x = F(x_2^*)\Delta x.$$

Continuing in this fashion and putting the pieces together, the triple integral is approximately the Riemann sum

$$\sum_{i} F(x_i^*)\Delta x,$$

and the exact answer is

$$\int_{x=a_1}^{x=a_2} F(x)\ dx = \int_{x=a_1}^{x=a_2} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} \left[\int_{z=\psi_1(x,y)}^{z=\psi_1(x,y)} f(x,y,z)\ dz \right]\ dy \right]\ dx.$$

Application: Find mass of tetrahedron with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and mass density

$$x + y + z$$

at position (x,y,z). Solution: $\int\int\int x+y+z\ dxdydz$. To describe the region by inequalities, first describe the z interval given a fixed x and y, then describe the y interval given a fixed x in the xy plane, then describe the x interval on the x axis. See page 991.

Work out #24 – see notes. Do not evaluate!

Section 16.8: Cylindrical Coordinates

Express x and y in terms of polar coordinates. Assume that the region $a_1 \leq x \leq a_2$, $\phi_1(x) \leq y \leq \phi_2(x)$, $\psi_1(x,y) \leq z \leq \psi_2(x,y)$ can be expressed in the form $a \leq \theta \leq b$, $\phi_1(\theta) \leq r \leq \phi_2(\theta)$, $\psi_1(x,y) \leq z \leq \psi_2(x,y)$. Then we can view the triple integral formula above as a double integral $dxdy$ where the integrand is expressed as an integral dz. So we use the polar coordinates formula to write

$$\int_{x=a_1}^{x=a_2} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} \left[\int_{z=\psi_1(x,y)}^{z=\psi_1(x,y)} f(x,y,z)\ dz \right]\ dy \right]\ dx =$$

$$\int_{\theta=a}^{\theta=b} \left[\int_{r=\phi_1(\theta)}^{r=\phi_2(\theta)} \left[\int_{z=\psi_1(r\cos\theta, r\sin\theta)}^{z=\psi_1(r\cos\theta, r\sin\theta)} rf(r\cos\theta, r\sin\theta, z)\ dz \right]\ dr \right]\ d\theta.$$

Work out #20 – see notes.