Week ??? Lectures

Sections 12.5 and 12.6

Section 12.5: Triple Integrals

Example: Find mass of a solid region with mass density function $f(x, y, z)$ kg/cubic meter at position (x, y, z).

Solution: Cut region into rectangular solids of dimension $\Delta x \Delta y \Delta z$. Go to i^{th} solid and multiply volume by mass density $f(x_i^*, y_i^*, z_i^*)$. Add. The limit of these expressions is denoted by the symbol $\int\int\int_R f(x, y, z)\ dV$.

Evaluation: Assume function being integrated is $f(x, y, z)$, over region $a_1 \leq x \leq a_2$, $\phi_1(x) \leq y \leq \phi_2(x)$, $\psi_1(x, y) \leq z \leq \psi_2(x, y)$. To visualize this region, first draw a plane parallel to the xz plane through $x = a_1$ and $x = a_2$. The region is between these boundary planes. Now consider any x_0 in the interval $[a_1, a_2]$. Then y and z satisfy $\phi_1(x_0) \leq y \leq \phi_2(x_0)$, $\psi_1(x_0, y) \leq z \leq \psi_2(x_0, y)$. Therefore y is varying between two numbers and, for any particular value of y, z is varying between two boundary curves. So this looks like a type I region with y along the horizontal axis and z along the vertical axis.

To approximate the mass, dissect the interval $[a_1, a_2]$ into $\{x_0, x_1, \ldots, x_n\}$. Find approximate mass of the region between x_{i-1} and x_i. Pick an arbitrary value $x_i^* \in [x_{i-1}, x_i]$. We will approximate the slice of the region through $x = x_{i-1}$ and $x = x_i$ as being described by $x_{i-1} \leq x \leq x_i$, $\phi_1(x_i^*) \leq y \leq \phi_2(x_i^*)$, and $\psi_1(x_i^*, y) \leq z \leq \psi_2(x_i^*, y)$. Consider the yz slice through $x = x_i^*$. Chop this up into rectangles, pick an arbitrary (y_j^*, z_j^*) in the j^{th} rectangle, and treat the mass density in the rectangular solid defined by $[x_{i-1}, x_i]$ and the j^{th} rectangle as being the constant $f(x_i^*, y_j^*, z_j^*)$. Then the approximate mass of this rectangular solid is $f(x_i^*, y_j^*, z_j^*)\Delta x \Delta y \Delta z$. Add up the masses of the rectangular solids to obtain

$$\left(\sum_j f(x_i^*, y_j^*, z_j^*)\Delta y \Delta z \right) \Delta x.$$

The expression inside parentheses looks like the Riemann sum for the type I double integral

$$\int_{\phi_1(x_i^*)}^{\phi_2(x_i^*)} \int_{\psi_1(x_i^*,y)}^{\psi_2(x_i^*,y)} f(x_i^*, y, z)\ dz\ dy.$$

So approximate the mass of the slice by $F(x_i^*)$, where

$$F(x) = \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\ dz\ dy.$$

So the total approximate mass is

$$\sum_i F(x_i^*) \Delta x.$$

This is a Riemann sum for the integral

$$\int_{a_1}^{a_2} F(x)\ dx = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\ dz\ dy\ dx.$$

Fubini's theorem says that if the region of integration is a rectangular solid, you can integrate in any order. Otherwise, you must be careful to integrate in the order dictated by the shape of the region.

More generally, try to describe the region as being bounded by two parallel planes. Then describe parallel slices as type I regions.

Application: Find mass of tetrahedron with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and mass density

$$x + y + z$$

at position (x, y, z).

Do some problems out of the book.

Note that a volume can be considered mass with mass density 1. Also, compute center of mass.

Section 12.6: Triple Integrals in Cylindrical Coordinates

In the expression

$$\int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\ dz\ dy\ dx,$$

let

$$F(x, y) = \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\ dz.$$

The we are evaluating

$$\int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} F(x,y)\ dy\ dx.$$

If possible, evaluate this using polar coordinates:

$$\int_{\theta_1}^{\theta_2} \int_{f(\theta)}^{g(\theta)} rF(r\cos\theta, r\sin\theta)\ dr\ d\theta.$$

This yields

$$\int_{\theta_1}^{\theta_2} \int_{f(\theta)}^{g(\theta)} \int_{\psi_1(r\cos\theta, r\sin\theta)}^{\psi_2(r\cos\theta, r\sin\theta)} rf(r\cos\theta, r\sin\theta, z)\ dz\ dr\ d\theta.$$

Work out some examples.