

Section 15.4: The Gradient as a Normal; Tangent Lines and Tangent Planes

Review of lines: The general equation of a line passing through (x_0, y_0) is $a(x - x_0) + b(y - y_0) = 0$. This says that the vector (a, b) is normal to the vector $(x - x_0, y - y_0)$, which can be viewed as a displacement along the direction of the line. A perpendicular line will have perpendicular normal. We can take the perpendicular normal to be $(-b, a)$. Therefore the line perpendicular $a(x - x_0) + b(y - y_0) = 0$ and passing through (x_0, y_0) can be taken to be $-b(x - x_0) + a(y - y_0) = 0$.

Consider our temperature example $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ defined by $f(x, y) = x^2 + 4y^2 + 10$. Temperature at position $(2, 3)$ is 50. This point lies on the level curve $f(x, y) = 50$, i.e. $x^2 + 4y^2 = 40$, which is the equation of an ellipse. We will find the equation of the tangent line and normal line to the level curve at this point, and develop a general formula for functions of 2 variables.

Strategy: show that the gradient is normal to the curve. Use gradient to define tangent line, then use a vector perpendicular to the gradient to define normal line.

To prove that the gradient is normal to the curve, let $r(t) = (x(t), y(t))$ be any curve which satisfies $x(t)^2 + 4y(t)^2 = 40$ and $r(0) = (2, 3)$. Many ways to do this, but the relevant point is that $f(r(t)) = c$.

Next, differentiate this formula using the chain rule: $\nabla f(r(t)) \cdot r'(t) = 0$.

Next, evaluate at $t = 0$: $\nabla f(2, 3) \cdot r'(0) = 0$.

Since $r'(0)$ is parallel to the curve and the dot product is 0, $\nabla f(2, 3)$ is perpendicular to $r'(0)$, hence perpendicular to the curve.

Hence tangent line equation is $f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3) = 0$ and normal line equation is $-f_y(2, 3)(x - 2) + f_x(2, 3)(y - 3) = 0$.

Review of planes: The equation of a plane through (x_0, y_0, z_0) and normal to (a, b, c) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Tangent plane to surface in 3 dimensions at P : plane containing all $r'(0)$ for $r(t)$ lying on surface and $r(0) = P$.

Normal line: in direction of normal to plane through this point.

Example: find plane tangent to the sphere $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$.

Method: There are an infinite number of space curves $r(t)$ which travel on the surface of this sphere and satisfying $r(0) = (1, 1, 1)$, and in each case $r'(0)$ can be viewed as a displacement in the tangent plane. We need to find a vector normal to all these tangent vectors. To do this, set

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Then the sphere can be viewed as the level surface corresponding to $c = 3$. Any trajectory $r(t)$ satisfies

$$x(t)^2 + y(t)^2 + z(t)^2 = 3,$$

i.e.

$$f(r(t)) = 3.$$

Differentiating, we obtain

$$\nabla f(r(t)) \cdot r'(t) = 0.$$

At $t = 0$ we have $\nabla f(1, 1, 1) \cdot r'(0) = 0$, hence $\nabla f(1, 1, 1)$ is the normal we are looking for. Hence the tangent plane has equation

$$f_x(1, 1, 1)(x - 1) + f_y(1, 1, 1)(y - 1) + f_z(1, 1, 1)(z - 1) = 0.$$

Note that the normal line equation is the equation of the line through $(1, 1, 1)$ with direction vector $\nabla f(1, 1, 1)$.

Note that one can find angle any line creates with tangent plane at intersection with sphere. Use dot product to find angle with normal, then find complementary angle.

Tangent plane to the graph of $f : \mathbf{R}^2 \rightarrow \mathbf{R}$: Treat as $f(x, y) - z = 0$. If we define

$$g(x, y, z) = f(x, y) - z,$$

then the graph is a level surface corresponding to $c = 0$. Example: find equation of plane tangent to graph of $f(x, y) = x^2 + 4y^2 + 10$ at the point $(2, 3, 50)$. Then find normal line through this point.

Section 15.5: Local Extreme Values

Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be given. c is a local maximum output value of f , and (x_0, y_0) is the input value producing c , if $f(x, y) \geq c$ for all (x, y) in a neighborhood of c . Note that this requires (x_0, y_0) to belong to the interior of the domain of f .

Example: $f(x, y) = (x - 1)^2 + (y - 2)^2 + 5$. Then $c = 5$ and $(x_0, y_0) = (1, 2)$.

Example: $f(x, y) = (x - 1)^2 - (y - 2)^2 + 5$. Then $(1, 2)$ is not the location of an extreme value: consider inputs of the form $(1, 2) + \lambda(1, m)$ with $m < 1$ and $m > 1$. Also, in $x = 1$ plane we have the graph $z = 5 - (y - 2)^2$, namely parabola opening down. But in $y = 2$ plane we have the graph $z = 5 + (x - 1)^2$, parabola opening up.

Necessary condition for (x_0, y_0) to be location of an extreme value: Tangent to $r(t)$ traveling in graph and satisfying $r(0) = (x_0, y_0)$ must be horizontal. This implies that the normal to the plane points in the z direction. This implies that $(f_x(x_0, y_0), f_y(x_0, y_0), -1) = (0, 0, -1)$, which implies that $\nabla f(x_0, y_0) = 0$.

Another way to prove this: $f(r(t))$ has local max or min at $r = 0$, therefore derivative is zero. Implies $\nabla f(r(0)) \cdot r'(0) = 0$ for all $r(t)$. Choosing $r'(0)$ in direction of gradient says length of gradient vector is zero. Therefore gradient is zero. This allows us to generalize to higher dimensions.

Definition: (x_0, y_0) is stationary point of f .

Look at examples. Any extreme values will occur at location of stationary points. But not all stationary points are locations of extreme values. These are called saddle points. So we need additional information to decide which of the stationary points are locations of extreme values and not saddle points.

Method: Second Partial test. Note that no information is given in rule 2 when $A = 0$. See page 906 for statement of test. No proof given, but consider

$$f(x, y) = a(x - 1)^2 + c(y - 2)^2.$$

It works!

Section 15.6: Absolute Extreme Values

Extreme Value of a function: the largest or smallest output value (global, not local).

Extreme Value Theorem: page 912. We've seen this already.

How to find extreme values: First inspect the local extreme values. This requires that you find stationary points in the interior of the domain, because computing ∇f requires a limit calculation and limits are defined at interior points.

Next, check boundary points. Compare.

Example: Temperature function $f(x, y) = x^2 + 4y^2 + 10$ defined on square of length 1 with vertex at origin and sitting in first quadrant. There are no stationary points in the interior. So the maximum temperature is going to occur on the boundary somewhere. Parameterize the sides of the square.

Example: Temperature function on circle $x^2 + y^2 = 100$. Stationary point is in interior. Parameterize the circle as $r(t) = (10 \cos t, 10 \sin t)$. Then $F(t) = 100 \cos^2 t + 400 \sin^2 t + 10 = 410 - 300 \sin^2 t$, $F'(t) = -600 \sin t \cos t$, so one of these is zero. Look at the four points of the circle. Max on y axis, min on x axis.

Look at this example on a triangle. Parameterize the lines using point and direction vector.