Math 223 Week 6 Lectures

Sections 12.1 and 12.2

Section 12.1: Double Integrals over Rectangles

Review the definite integral in one variable. Do an example of mass density of a cable (average mass per inch, or limit of mass of a small piece divided by length of small piece).

Let $f(x,y) = 10 + x^2 + 4y^2$ represent the mass density of a metal place which extends over the region $[0,5] \times [0,8]$. What is the total mass of the plate? (Mass density is found by taking the limit of the mass within a square rectangle divided by the area of that rectangle, so units will be kg per meter squared.)

We can get an approximate solution: slice the rectangle into small sub-rectangles. Mass density does not vary much within a small rectangle. Measure mass density anywhere inside the rectangle, then multiply by area of rectangle to get estimate of mass of that rectangle. Add the results together. Do it.

Double integral: defined on page 665.

Why not generalize this to more general regions. Dissect regions into little regions, do the sum, take limit as the area of the largest region approaches 0.

Note: If you represent mass density by a z-coordinate, you are obtaining volumes of rectangular cubes. So another interpretation is volume.

Suppose mass density is $f(x,y) = \sqrt{1 - x^2 + y^2}$ within the circle of radius 1. Then $\int\int \sqrt{1 - x^2 - y^2}\ dA$ is one-half the volume of a sphere of radius 1, namely $\frac{2}{3}\pi$.

Iterated integrals over a rectangle: compute the mass of the plate above. Slice the x-axis into little strips. Pick the i^{th} strip. Slice this vertically, obtaining the coordinates (x_i^*, y_0), (x_i^*, y_1),..., (x_i^*, y_n). Approximate mass

$$\sum_j f(x_i^*, y_j^*)(x_i - x_{i-1})(y_j - y_{j-1}) \to \int_{y_0}^{y_n} f(x_i^*, y)(x_i - x_{i-1})\ dy.$$

Now let $F(x) = \int_{y_0}^{y_n} f(x,y)\ dy$ and call this answer $F(x_i^*)(x_i - x_{i-1})$. So total mass is approximately

$$\sum_i F(x_i^*)(x_i - x_{i-1}) \to \int_{x_0}^{x_n} F(x)\ dx = \int_{x_0}^{x_n} \int_{y_0}^{y_n} f(x,y)\ dy\ dx.$$

Get the same result by different dissection. See Fubini's Theorem, page 670. Proof is omitted.

Now do some volume problems from the exercises.

Do problem 41. Get different answers of $1/over2$ and $-\frac{1}{2}$. Problem: not continuous at $(0,0)$ – approach along $y = mx$. Note that in computing the integrals, get improper integrals.

Section 12.2: Double Integrals over General Regions

Type I and Type II regions. See pp. 675,676. Logic: same as volume calculation above. One must decide which one to use to compute volume, based on the shape of the region. Essential to draw it!

Changing the order of integration: sketch region implied, then use perpendicular slices. See problems 31–36 and 37–42.

A good one: problem 52.