Math 223 Week 5 Lectures: Sections 11.7, 11.8

Section 11.7: Maximum and Minimum Values

Local maximum value and local minimum value of $z = f(x, y)$. There must be local minimum holding one variable fixed, so both partial derivatives are zero. This location called critical point of function.

Example: $f(x, y) = (x - 1)^2 + (y - 2)^2 + 5$ at $(1, 2)$.

Note: there's a min but no max. There may be neither. Example: $f(x, y) = (x - 1)^2 - (y - 2)^2 + 5$. Consider just changing x or just changing y. This point is called saddle point.

Method: Second Derivative test, page 646. Consider

$$f(x, y) = p(x - 1)^2 + q(y - 2)^2 + 5.$$

It works! Note also that no information is given when $D = 0$. Consider $f(x, y) = px^4 + qy^4$. Can get a local max, a local min, or a saddle, depending on p or q.

Extreme Value of a function: the largest or smallest output value (global, not local).

Extreme Value Theorem: page 649.

How to find extreme values: First inspect the local extreme values. This requires that you find stationary points in the interior of the domain, because computing ∇f requires a limit calculation and limits are defined at interior points.

Next, check boundary points. Compare.

Example: Temperature function $f(x, y) = x^2 + 4y^2 + 10x$ defined on square of length 1 with vertex at origin and sitting in first quadrant. There are no stationary points in the interior. So the maximum temperature is going to occur on the boundary somewhere. Parameterize the sides of the square.

Example: Temperature function on and within circle $x^2 + y^2 = 100$. Stationary point is in interior. Parameterize the circle as $r(\theta) = (10 \cos \theta, 10 \sin \theta)$. Then $F(\theta) = 100 \cos^2 \theta + 400 \sin^2 \theta + 10 \cos \theta = 400 - 300 \sin^2 \theta + 10 \cos \theta$, $F'(\theta) = -600 \sin \theta \cos \theta - 10 \sin \theta$. Solving $F'(\theta) = 0$ determines four points of the circle. Compare.

Do some word problems. For example, find minimum distance between $y = x^2$ and $y = x - 1$. Minimize $f(p, q) =$ distance between (p, p^2) and $(q, q-1)$. More generally, two arbitrary curves.

Section 11.8: Lagrange Multipliers

Problem: Maximize $f(x, y)$ subject to $g(x, y) = c$.

Solution: Let C be the curve defined by $g(x, y) = c$. Let $(x_0, y_0) \in C$ be the location of a local maximum of $f(x, y)$ when restricted to points in C. Let $r(t)$ be a trajectory in C which satisfies $r(0) = (x_0, y_0)$. Then $f(r(t))$ has a local maximum at $t = 0$, therefore $\nabla f(r(0)) \cdot r'(0) = 0$. Also, we have $g(r(t)) = c$ for all t, therefore $\nabla g(r(0)) \cdot r'(0) = 0$. This means that both $\nabla f(r(0))$ and $\nabla g(r(0))$ are perpendicular to $r'(0)$. Hence they are parallel to each other. Hence

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some λ. We should be able to solve for λ and (x_0, y_0) using the two equations

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \text{ and } g(x_0, y_0) = c.$$

λ is called a Lagrange Multiplier.

Example: Minimize $x^2 + y^2$ on the hyperbola $xy = 1$.

Solution: The function is $f(x, y) = x^2 + y^2$. The hyperbola is a level curve of $g(x, y) = xy$. The stationary points (x_0, y_0) satisfy

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

$$(2x_0, 2y_0) = \lambda(y_0, x_0),$$

$$2x_0 = \lambda y_0 \qquad 2y_0 = \lambda x_0,$$

hence $(x_0, y_0) = (\pm 1, \pm 1)$. Therefore the minimum value of $x^2 + y^2$ is 2.

Example: Maximize rectangular solid volume subject to surface area $6a^2$.

Solution: Maximize $f(x, y, z) = xyz$ on the level curve of $g(x, y, z) = 2xy + 2xz + 2yz$ corresponding to $c = 6a^2$. Solve for λ, get three equal solutions. They imply $x = y = z = a$.

Maximize $f(x, y, z)$ subject to $g(x, y, z) = c$ and $h(x, y, z) = d$: Let $r(t)$ lie in both level surfaces. Want $\nabla f \cdot r'(0) = 0$, $\nabla g \cdot r'(0) = 0$, and $\nabla h \cdot r'(0) = 0$. So

$r'(0)$ is perpendicular to the g and h gradients, so can be taken to be normal to these gradients, so normal to all linear combinations (the plane defined by them). This places the gradient of f in this plane, so $\nabla f = \lambda \nabla g + \mu \nabla h$.

Work out some problems.