Math 223 Week 4 Lectures: Sections 11.5, 11.6

Section 11.5: The Chain Rule

Temperature on metal plate: $T(x, y) = 10 + x^2 + 4y^2$. Temperature at position $(2, 5)$: $T(2, 5) = 114$. What direction does bug walk in initially to drop temperature as fast as possible in degrees per foot?

Head h units in direction $(\cos\theta, \sin\theta)$. Change in temperature is approximately $T_x(2, 5)h\cos\theta + T_y(2, 5)h\sin\theta$ degrees in h feet, hence rate is

$$T_x(2, 5)\cos\theta + T_y(2, 5)\sin\theta$$

degrees per foot. As a dot product,

$$\frac{dT}{dh} = (T_x, T_y) \cdot (\cos\theta, \sin\theta).$$

Call this $R(\theta)$. $R(\theta)$ is optimized when $R'(\theta) = 0$, $(T_x, T_y) \cdot (-\sin\theta, \cos\theta) = 0$, i.e. when (T_x, T_y) and $(-\sin\theta, \cos\theta)$ are perpendicular, i.e. when $(T_x, T_y) = \lambda(\cos\theta, \sin\theta) = \sqrt{T_x^2 + T_y^2}(\cos\theta, \sin\theta)$. Hence we get $\tan\theta = \frac{T_x}{T_y}$, $\theta = \tan^{-1}\frac{T_x}{T_y}$.

Incidentally, $R(\theta) = 0$ when we use $(T_x, T_y) \cdot (\cos\theta, \sin\theta) = 0$, $(T_x, T_y) = \lambda(-\sin\theta, \cos\theta)$, $\theta = -\tan^{-1}\frac{T_y}{T_x}$.

Note also that $(T_x, T_y) = \sqrt{T_x^2 + T_y^2}(-\sin\theta, \cos\theta)$. So the direction of no change (tangent direction) is perpendicular to the direction max and min change.

New question: suppose the bug walks along the parabolic curve $r(t) = (x(t), y(t)) = (t, t^2)$. How is the rate of change of temperature along this curve varying with respect to time? For example, as the bug passes through position $(2, 4)$?

Change in temperature from position $r(t)$ to $r(t + h)$ is $T_x dx + T_y dy$. Note $dx = x(t + h) - x(t) \approx hx'(t)$ and $dy = y(t + h) - y(t) \approx hy'(t)$, so $\Delta T \approx T_x hx'(t) + T_y hy'(t)$. Moreover, change in position is approximately $h\sqrt{x'(t)^2 + y'(t)^2}$. Therefore rate of change of temperature is approximately

$$R(t) = \frac{T_x x'(t) + T_y y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} = \frac{(T_x, T_y) \cdot r'(t)}{|r'(t)|} = (T_x, T_y) \cdot T(t).$$

At what position is temperature changing at a rate of 10 degrees per second? Answer: Can get an equation for t, but need to approximate the answer.

One can ask the same question about rate in terms of degrees per foot. By a unit analysis, degrees per foot is degrees per second divided by feet per second, so

$$R(s) = \frac{R(t)}{|r'(t)|} = \frac{(T_x, T_y) \cdot r'(t)}{|r'(t)|^2}.$$

This is doable.

A general question: Given $z = f(x, y)$ and a path $r(t) = (x(t), y(t))$, let $F(t) = f(r(t))$ and compute $F'(t)$. Answer: move from position $(x(t), y(t)$ to position approximately $(x(t) + hx'(t), y(t) + hy'(t))$ from time t to time $t + h$. $\Delta F \approx f_x(r(t))hx'(t) + f_y(r(t))hy'(t)$, therefore $\frac{dF}{dt} \approx (f_x, f_y) \cdot r'(t)$. This gives rise to the chain rule, Case 1, page 625.

Another question: given $f(x, y)$ and $R(s, t) = (x(s, t), y(s, t))$, let $F(s, t) = f(R(s, t))$. What are F_s and F_t? Fix t and let s increase to $s + h$, then compute $\frac{df}{h}$. Note that $ds = h$ and $dt = 0$, therefore $dx = x_s ds + x_t dt = x_s h$ and $dy = y_s ds + y_t dt = y_s h$, therefore $df = f_x dx + f_y dy = f_x x_s h + f_y y_s h$, therefore $F_s = \frac{df}{h} = f_x x_s + f_y y_s$. This gives the chain rule, Case 2, page 627. We can compute F_t similarly.

Implicit differentiation: Let $F(x, y) = c$ define y in terms of x. Find $\frac{dy}{dx}$.

Solution: As x increases to $x + dx$, y increases to $y + dy$. In the mean time, $dF = 0$ since we remain on the level curve of F. So we have

$$0 = dF = F_x dx + F_y dy,$$

hence

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Now go back and find the tangent and normal to the isothermal line 114 degrees.

Section 11.6: Directional Derivatives and the Gradient Vector

The gradient vector of the function $f(x, y)$ is (f_x, f_y). The directional derivative of f in the direction of the unit vector $u = (u_1, u_2)$ at position (x, y) is the rate of change of f along the path $r(t) = (x, y) + t(u_1, u_2)$ at time $t = 0$. As we have seen,

$$\frac{df}{dt} = \frac{f_x dx + f_y dy}{dt} = \frac{f_x t u_1 + f_y t u_2}{t} = f_x u_1 + f_y u_2 = (f_x, f_y) \cdot (u_1, u_2).$$

Hence we say

$$D_u f(x, y) = (f_x, f_y) \cdot u.$$

To compute tangent vector, we set $(f_x, f_y) \cdot u = 0$ to get $u = \frac{(-f_y, f_x)}{\sqrt{f_x^2 + f_y^2}}$. To maximize or minimize $D_u f$, we maximize $|(f_x, f_y)||u| \cos\theta$, where θ is the angle between them. We get a maximum when $\theta = 0$ and a minimum when $\theta = \pi$. So for maximum we have $u = \frac{(f_x, f_y)}{\sqrt{f_x^2 + f_y^2}}$ and for minimum we have $u = -\frac{(f_x, f_y)}{\sqrt{f_x^2 + f_y^2}}$.

Tangent plane and normal line to $F(x, y, z) = c$ at (a, b, c): Imagine a bug traveling along the path $r(t) = (x(t), y(t), z(t))$ passing through the point (a, b, c) at time zero. It's direction of travel will be in the plane tangent to this surface. So $r'(0)$ will lie in the plane, assuming it has one endpoint at (a, b, c). Let's compute $r'(0)$:

As t varies from 0 to dt, x varies from a to $a + dx = a + x'(0)dt$, y varies from b to $b + dy = b + y'(0)dt$, z varies from c to $c + dz = c + z'(0)dt$. Moreover, temperature does not vary, so $\frac{dF}{dt} = 0$. In other words,

$$0 = F_x dx + F_y dy + F_z dz = F_x x'(0)dt + F_y y'(0)dt + F_z z'(0)dt = 0.$$

This says $(F_x, F_y, F_z) \cdot r'(0) = 0$. So the gradient is normal to the tangent plane. Hence the tangent plane equation and the normal line equation, page 640.