Math 223 Week 2 Lectures: Sections 10.9, 11.1, 11.2

**Section 10.9: Motion in Space: Velocity and Acceleration**

Position vector: $r(t)$

Velocity vector: $v(t) = r'(t)$

Acceleration vector: $a(t) = v'(t) = r''(t)$

Examples: $r(t) = (\cos t, \sin t, 0)$, $r(t) = (\cos t, \sin t, \frac{t}{\pi})$, $r(t) = (\cos t, \sin t, \frac{t^2}{\pi^2})$.

Expressing velocity vector and acceleration vector in terms of $T$ and $N$:

$$v(t) = r'(t) = ||r'(t)||T(t) = \frac{ds}{dt}T(t)$$

$$a(t) = v'(t) = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T'(t) = \frac{d^2s}{dt^2}T + \frac{ds}{dt}||T'(t)||N(t) =$$

$$\frac{d^2s}{dt^2}T + \frac{ds}{dt}T'(t) = \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)^2 \kappa N(t).$$

We write $a_T = \frac{d^2s}{dt^2}$ and $a_N = \left(\frac{ds}{dt}\right)^2 \kappa$.

Computing $a_T$ efficiently:

$$a_T T + a_N N = a$$

$$a_T(T \cdot T) = a \cdot T$$

$$a_T = \frac{r''(t) \cdot r'(t)}{\frac{ds}{dt}}.$$

Computing $a_N$ efficiently:

$$a_T T + a_N N = a$$

$$a_N(N \times T) = a \times T$$

$$a_N = ||a \times T||$$

$$a_N = \frac{||r''(t) \times r'(t)||}{\frac{ds}{dt}}$$

When dealing with projectiles, balls, bullets, guns, etc, pretend that motion occurs in the $xy$ plane with acceleration $-gj$ where $g$ is 9.8 meters per second squared or 16.0 feet per second squared. Now do problems 19 and 21, page 587.

If a particle is moving in 3-space such a way that its acceleration is always toward the origin with magnitude inversely proportional to its distance from the origin, then it must be moving in an elliptical orbit in a plane through the origin. To see this, write $r''(t) = -\frac{\alpha}{||r(t)||^3} r(t)$. Then $r''(t)$ is parallel to $r(t)$, so $r(t) \times r''(t) = 0$. Therefore $\frac{d}{dt}(r(t) \times r'(t)) = 0$, therefore $r(t) \times r'(t) = h$ for some constant vector $h$. So $h$ is at all times perpendicular to $r(t)$ and $r(t)$ is in the plane normal to $h$. We will now choose a coordinate system so that $r(t) = |r(t)|(\cos\theta(t), \sin\theta(t), 0)$ and $r''(t) = -\frac{\alpha}{|r(t)|^2}(\cos\theta(t), \sin\theta(t), 0)$ and $h = (0, 0, |r(t)|^2\theta'(t))$.

We have

$$(r' \times h)' = r'' \times h + r' \times h' = r'' \times h = (-\alpha \sin\theta\theta', \alpha\cos\theta\theta', 0) =$$

$$(\alpha\cos\theta, \alpha\sin\theta, 0)',$$

therefore

$$r' \times h = (\alpha\cos\theta, \alpha\sin\theta, 0) + C = \frac{\alpha}{|r(t)|} r + C.$$

Hence

$$r \cdot (r' \times h) = \frac{\alpha}{|r|}|r|^2 + C \cdot r = \alpha|r| + c|r|\cos\psi.$$

By properties of the dot and cross product,

$$r \cdot (r' \times h) = (r \times r') \cdot h = h \cdot h = |h|^2.$$

Therefore

$$|h|^2 = \alpha|r| + c|r|\cos\psi.$$

This is the polar equation of an ellipse (one focus at origin, other on axis perpendicular to $C$).

Moreover, equal areas are swept out in equal time periods. Reason: Let $A(t)$ = area swept out after $t$ units of time. Then $A(t+h) - A(t)$ is approximately equal to $\frac{1}{2}||r(t) \times r(t+h)||$. Hence

$$\frac{A(t+h) - A(t)}{h} \approx \frac{1}{2}||r(t) \times \frac{r(t+h) - r(t)}{h}||$$

Let $h \to 0$. Obtain

$$A'(t) = \frac{1}{2}||r(t) \times r'(t)|| = \frac{1}{2}||h||,$$

which is a constant rate.

**Section 11.1: Functions of Several Variables**

Function notation, domain, range

The graph of a function $f : \mathbf{R}^2 \to \mathbf{R}$ is the set of all points $(x, y, z)$ where $z = f(x, y)$.

Graphing technique for $z = f(x, y)$: project graph at height $c$ onto $xy$ plane, and label the level curve by $c$. Then lift level curves up by $c$.

Example: $f(x, y) = x^2 + 4y^2 + 10$

Perhaps $x$ and $y$ have units of feet and $z$ has units of degrees, and $f(x, y)$ measures temperature at location $(x, y)$

Level curves are ellipses

Graph is Elliptic Paraboloid

Level surfaces: same idea for higher dimensions.

**Section 11.2: Limits and Continuity**

Objective: define $\lim_{x \to x_0} f(x) = L$. Intuitively, the distance between $f(x)$ and $L$, which is measured as $|f(x) - L|$, approaches 0 as the distance between $x$ and $x_0$, measured as $||x - x_0||$, approaches zero.

Note: $f(x)$ does not need to be defined at $x_0$. Example: $f(x, y) = \frac{\sin(x)}{x} + y$. Limit as $(x, y)$ approaches $(0, 5)$ is 5.

Example: $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$ doesn't exist. Consider letting $(x, y)$ approach the origin in the direction $(1, 1)$: limit is $\frac{1}{2}$. But in the direction $(1, 2)$: $\frac{2}{5}$. No consistent limit. So at the very least we want the same result coming in along any line.

Example: $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$ doesn't exist. Along direction $(1, m)$ where $m \neq 0$: approaches 0. Along the $x$-axis: approaches 0. But along the path $(t, t^2)$: approaches $\frac{2}{5}$.

Definition: the limit is $L$ if we are able to say the following: for each $\epsilon > 0$ we can find $\delta > 0$ such that whenever $x$ is within $\delta$ units of $x_0$, $f(x)$ is within $\epsilon$ units of $L$.

Example: $f(x, y) = 2x + 2y + 1$. Limit as $(x, y)$ approaches $(0, 0)$ is 1. Proof: Let $\epsilon = .1$ be given. If $x^2 + y^2 < \delta^2$ then $-\delta < x < \delta$ and $-\delta < y < \delta$, therefore $-4\delta < 2x + 2y < 4\delta$, so we should pick $\delta = \frac{.1}{4} = .025$. More generally, we should pick $\delta = \frac{\epsilon}{4}$.

Function continuous at a point: limit can be evaluated by direct substitution.

Continuity on a set: continuous at all points of set.

Continuity properties: projections, sums, products, compositions are all continuous.

Rational polynomial functions are continuous where denominator is not zero.

The function $f(x, y) = \frac{\sin(x)}{x} + y$ where $x \neq 0$, $f(x, y) = y$ where $x = 0$ is not continuous at $(0, 1)$. However, we can modify this example to produce a continuous functions.

Use of continuity: Extreme Value Theorem, page 912.

Consider $f(x) = \frac{1}{x^2}$ where $x \neq 0$ and $f(0) = 5$. On the interval $[-1, 1]$ there is an absolute minimum but no absolute maximum.

Clairaut's Theorem p. 613: continuity of first partials and mixed second partials on an open set $U$ guarantees equality of mixed partials on $U$. (Define open set.)