Sequences and Series of Functions (Rudin)

Stone-Weierstrass Theorem: Let \( f : [a, b] \to \mathbb{R} \) be continuous. Then there exists a sequence of polynomials \((P_n(x))\) such that converges uniformly to \( f \) on \([a, b]\).

**Proof:** Consider any continuous function \( g : \mathbb{R} \to \mathbb{R} \) that satisfies \( g(x) = 0 \) for \( x \not\in [0, 1] \). For each \( n \geq 1 \) set

\[
Q_n(x) = \frac{(1 - x^2)^n}{\int_{-1}^{1} (1 - x^2)^n \, dx}.
\]

The area under the curve \( y = Q_n(x) \) over \([-1, 1]\) is equal to 1, and has the shape of a bell curve with most of it’s area concentrated in a narrow band above \( x = 0 \). The polynomials \( Q_{10}(x) \), \( Q_{20}(x) \), \( Q_{30}(x) \) are plotted below:

For \( 0 \leq x \leq 1 \) define

\[
P_n(x) = \int_{-1}^{1} g(x + t)Q_n(t) \, dt.
\]

\( P_n(x) \) is a polynomial of degree \( \leq 2n \):

\[
\int_{-1}^{1} g(x + t)t^k \, dt = \int_{x-1}^{x+1} g(u)(u-x)^k \, du = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^i \int_{x-1}^{x+1} g(u)u^{k-i} \, du.
\]

We have

\[
|P_n(x) - g(x)| \leq \int_{-1}^{1} |g(x + t) - g(x)|Q_n(t) \, dt.
\]
Let $M = \sup g(x)$. Since $g$ is uniformly continuous on $\mathbb{R}$, there exists $\delta_k > 0$ such that $|t| < \delta_k$ implies $|g(x + t) - g(x)| \leq \frac{1}{k}$ for all $x$. This yields

$$|P_n(x) - g(x)| \leq 2M \int_{-\delta_k}^{-\delta_k} Q_n(t) \, dt + \frac{1}{k} \int_{-\delta_k}^{\delta_k} Q_n(t) \, dt + 2M \int_{\delta_k}^{1} Q_n(t) \, dt.$$  

We have

$$\frac{1}{k} \int_{-\delta_k}^{\delta_k} Q_n(t) \, dt \leq \frac{1}{k}.$$  

We also have

$$\int_{-1}^{1} (1 - x^2)^n \, dx \geq 2 \int_{0}^{1} (1 - x)^n \, dx = \frac{2}{n + 1},$$

hence for $\delta_k \leq |x| \leq 1$ we have

$$|Q_n(x)| \leq \frac{n + 1}{2} (1 - \delta_k^2)^n.$$  

Therefore

$$|P_n(x) - g(x)| \leq 2M(n + 1)(1 - \delta_k^2)^n + \frac{1}{k}$$

for $0 \leq x \leq 1$, i.e.

$$||P_n - g|| \leq 2M(n + 1)(1 - \delta_k^2)^n + \frac{1}{k}.$$  

Let $\epsilon > 0$ be given. Choose $k$ so that $\frac{1}{k} < \frac{\epsilon}{2}$. For sufficiently large $n$, $2M(n + 1)(1 - \delta_k^2)^n < \frac{\epsilon}{2}$, hence $||P_n - g|| < \epsilon$. Therefore $P_n \to g$ uniformly on $[0, 1]$.

Given $f : [a, b] \to \mathbb{R}$, $g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$ satisfies $g(a) = g(b) = 0$ and $h(x) = g((b - a)x + a)$ satisfies $h(0) = h(1) = 0$. If $P_n(x) \to h(x)$ uniformly on $[0, 1]$, then $P_n(x) \to g((b - a)x + a)$ on $[0, 1]$, therefore $P_n(x - a/b) \to g(x)$ uniformly on $[a, b]$, therefore

$$P_n \left( \frac{x - a}{b - a} \right) + f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \to f(x)$$

uniformly on $[a, b]$. 

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**Corollary:** If \( f(0) = 0 \) and \( P_n(x) \to f \) uniformly on \([-a,a]\), then \( P_n(0) \to 0 \), hence \( P_n(x) - P_n(0) \to f \) uniformly on \([-a,a]\). So \( f \) can be uniformly approximated by a polynomial with zero constant term.

**Definition:** An algebra \( A \) of functions \( f : E \to \mathbb{R} \) is a set of functions closed under addition, multiplication, and scalar multiplication. An algebra \( A \) is said to separate points on \( E \) if for each \( x \neq y \) in \( E \) there exist \( f, g \in A \) such that \( f(x) \neq g(y) \), and to vanish at \( x \in E \) if \( f(x) = 0 \) for all \( f \in A \).

**Theorem:** Let \( A \) be an algebra of bounded functions from \( E \) to \( \mathbb{R} \). Let \( ||f|| = \sup_{x \in E} f(x) \). Then \( d(f, g) = ||f - g|| \) is a metric on \( A \) and \( \overline{A} \) is an algebra of bounded functions from \( E \) to \( \mathbb{R} \).

**Proof:** Clearly \( ||f - f|| = 0 \), \( f \neq g \Rightarrow ||f - g|| > 0 \), \( ||f - g|| = ||g - f|| \). Now let \( f, g, h \in A \) be given. For any \( x \in E \),

\[
|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f|| + ||g||,
\]

therefore \( ||f + g|| \leq ||f|| + ||g|| \). This implies

\[
||f - h|| = ||(f - g) + (g - h)|| \leq ||f - g|| + ||g - h||.
\]

Also, for any \( x \in E \),

\[
|f(x)g(x)| \leq ||f||||g(x)|| \leq ||f||||g||,
\]

therefore \( ||fg|| \leq ||f||||g|| \). Clearly \( ||cf|| = |c||||f|| \) for all \( c \in \mathbb{R} \). Now suppose \( f_n \to f \), \( g_n \to g \), and \( c \in \mathbb{R} \). Then

\[
||f_n + g_n - f - g|| = ||f_n - f|| + ||g_n - g|| \to 0,
\]

hence \( f_n + g_n \to f + g \). Also,

\[
||f_ng_n - fg|| \leq ||f_ng_n - f_ng|| + ||f_ng - fg|| \leq ||f_n||||g_n - g|| + ||f_n - f||||g|| \to 0,
\]

therefore \( f_ng_n \to fg \). Finally,

\[
||cf_n - cf|| = |c||||f_n - f|| \to 0,
\]

therefore \( cf_n \to cf \).

Given \( f_n \to f \), for any \( x \in E \) we have

\[
|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq ||f - f_n|| + ||f_n|| \leq 1 + ||f_n||
\]

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for \( n \geq n_0 \). Therefore \( ||f|| \leq 1 + ||f_{n_0}|| \).

**Lemma**: Let \( u \neq v \in \mathbb{R}^n \) and \( a, b \in \mathbb{R} \) be given. Then there exists \( p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \) such that \( p(u) = a \) and \( p(v) = b \).

**Proof**: Set \( g(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 + 1 \). Since \( u \neq v \), \( x_k(u) \neq x_k(v) \) for some \( k \). We can set

\[
p(x_1, \ldots, x_n) = \frac{aq(x_1, \ldots, x_n)(x_k - v_k)}{q(u_1, \ldots, u_n)(u_k - v_k)} + \frac{bq(x_1, \ldots, x_n)(x_k - u_k)}{q(u_1, \ldots, u_n)(v_k - u_k)}\]

**Theorem**: Let \( K \subseteq \mathbb{R}^n \) be a compact set. Then a function \( f : K \to \mathbb{R} \) is continuous if and only if there exists a sequence \( (f_n) \) in \( \mathbb{R}[x_1, \ldots, x_n] \) such that \( f_n \to f \) uniformly on \( K \).

**Proof**: Let \( \mathbb{A} \) be the set of polynomial functions on \( K \). Suppose \( f_n \to f \) uniformly, where each \( f_m \in \mathbb{A} \). Then \( f \) is continuous: Let \( \epsilon > 0 \) be given. Choose \( n \) so that \( ||f_n - f|| < \frac{\epsilon}{3} \). Since \( f_n \) is continuous on \( K \) and \( K \) is compact, \( f_n \) is uniformly continuous on \( K \), hence there exists \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \). Hence \( |x - y| < \delta \) implies

\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq ||f - f_n|| + \frac{\epsilon}{3} + ||f_n - f|| < \epsilon.
\]

Conversely, let \( f : K \to \mathbb{R} \) be continuous. We will show that \( f \in \overline{\mathbb{A}} \) as follows:

1. For all \( g \in \overline{\mathbb{A}} \), \( |g| \in \overline{\mathbb{A}} \). Proof: Let \( g \in \overline{\mathbb{A}} \) and \( \epsilon > 0 \) be given. Choose a polynomial \( P(x) \) with zero constant term such that \( ||P(x) - | \cdot || < \frac{\epsilon}{2} \) on \([-||g||, ||g||]\). Then \( ||P(g) - |g|| < \frac{\epsilon}{2} \). Given \( g_n \to g \), we have \( P(g_n) \to P(g) \). Choose \( n \) so that \( ||P(g_n) - P(g)|| < \frac{\epsilon}{2} \). Hence \( ||P(g_n) - |g|| < \epsilon \). Since \( P(g_n) \in \mathbb{A} \), \( |g| \in \overline{\mathbb{A}} \).

2. If \( g, h \in \mathbb{A} \), then \( \max(g, h) \in \overline{\mathbb{A}} \) and \( \min(g, h) \in \overline{\mathbb{A}} \). Proof:

\[
\max(g, h) = \frac{g + h}{2} + \frac{|g - h|}{2},
\]

\[
\min(g, h) = g + h - \max(g, h).
\]
3. Fix \( x \in K \). Then there exists \( f_x \in \overline{A} \) such that \( f_x(x) = f(x) \) and

\[
f_x(k) > f(k) - \epsilon
\]

for all \( k \in K \). Proof: For each \( y \in K \) choose \( g_y \in A \) such that \( g_y(x) = f(x) \) and \( g_y(y) = f(y) \). Then \( y \in (g_y - f)^{-1}(\epsilon, \infty) \), hence

\[
K = \bigcup_{y \in K} \{(g_y - f)^{-1}((-\epsilon, \infty)) : y \in K\},
\]

hence by compactness of \( K \) there exist \( y_1, \ldots, y_a \in K \) such that for each \( k \in K \) there exists \( i \) such that \( g_{y_i}(k) > f(k) - \epsilon \). We can set \( f_x = \max(g_{y_1}, \ldots, g_{y_a}) \). In particular, \( f_x(x) = x \).

4. We \( x \in (f_x - f)^{-1}((\infty, \epsilon)) \) for each \( x \in K \), hence

\[
K = \bigcup_{x \in K} \{(f_x - f)^{-1}((\infty, \epsilon)) : y \in K\},
\]

hence by compactness of \( K \) there exist \( x_1, \ldots, x_b \in K \) such that for each \( k \in K \) there exists \( i \) such that \( f_{x_i}(k) < f(k) + \epsilon \). Setting \( f_\epsilon = \min(f_{x_1}, \ldots, f_{x_b}) \in \overline{A} \), we have \(||f_\epsilon - f|| < \epsilon\). Since \( \epsilon \) is arbitrary, \( f \in \overline{A} \).