Selected Solutions to HW 9:

[1.5] $\tau(\sum_{i=0}^n a_i u^i) = \sum_{i=0}^n a_i(\tau(u))^i$, so τ is determined by $\tau(u)$.

[1.20] We are dealing with the extension $\mathbb{Q} \subseteq \mathbb{Q}[u_1, u_2]$ where $u_1 = 2^{1/4}$ and $u_2 = 2^{1/4}i$. So $\text{irr}(u_1, \mathbb{Q}) = \text{irr}(2^{1/4}, \mathbb{Q}) = x^4 - 2$ by Eisenstein's criterion ($p = 2$) and $\text{irr}(u_2, \mathbb{Q}[u_1]) = \text{irr}(2^{1/4}i, \mathbb{Q}[2^{1/4}]) = x^2 + (2^{1/4})^2$ since $2^{1/4}i \notin \mathbb{Q}[2^{1/4}]$. The roots of $x^4 - 2$ are all present in the extension field, so $v_1 = 2^{1/4}i$ is a valid choice. Given this choice we have $(x^2 + (2^{1/4})^2)' = x^2 + (2^{1/4}i)^2 = x^2 - 2^{1/2}$, which has both roots present in the extension field. So $v_2 = 2^{1/4}$ is a valid choice. Given these choices for v_1 and v_2, the Galois Group Algorithm constructs the automorphism $a(2^{1/4}, 2^{1/4}i) \mapsto a(2^{1/4}i, 2^{1/4})$. In particular, $2^{1/4} \mapsto 2^{1/4}i$ and $2^{1/4}i \mapsto 2^{1/4}$.

[2.6] Since f and g are the minimal polynomials of u and v over K respectively, $[K[u] : K] = n$ and $[K[v] : K] = m$. Since each of the latter numbers are divisors of $[K[u, v] : K]$ by the Tower Theorem, and since m and n are coprime, $mn | [K[u, v] : K]$. Therefore $[K[u, v] : K] \geq mn$. On the other hand, $[K[u, v] : K] = [K[u][v] : K[u]][K[u] : K]$, and since $\text{irr}(v, K[u]) | g(x)$ and $\text{irr}(u, K) = f(x)$, $[K[u, v] : K] \leq mn$. So in fact $[K[u, v] : K] = mn$. A bit of algebra yields $[K[u][v] : K[u]] = m$, which implies $g(x) = \text{irr}(v, K[u])$.