Selected Solutions to HW 5:

4.2: Let $r \neq 0$ in R. Then the sequence $r, r^2, r^3, \ldots$ must be finite because R is finite, and does not contain 0 since R is an integral domain. So at some point $r^i = r^j$ where $1 \leq i < j$. Hence $r^i(1 - r^{j-i}) = 0$, hence $1 - r^{j-i} = 0$, hence $r^{j-i} = 1$, hence $r(r^{j-i-1}) = 1$. In other words, r^{-1} exists.

4.20: The map $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(a, b) = a$ is a ring epimorphism with kernel $0 \times \mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain which is not a field, $0 \times \mathbb{Z}$ is a prime ideal which is not maximal.

4.24: (c) $I + \frac{2a}{2b+1} = I + \frac{0}{1}$ because $\frac{2a}{2b+1} - \frac{0}{1} = \frac{2a}{2b+1} \in I$. $I + \frac{2a+1}{2b+1} = I + \frac{1}{1}$ because $\frac{2a+1}{2b+1} - \frac{1}{1} = \frac{2a-2b}{2b+1} \in I$. This exhausts all possibility, so the two cosets boil down to $I + 0$ and $I + 1$. These must be unequal because $1 \notin I$ ($1 \in I \implies 1 = \frac{2a}{2b+1}$, a contradiction). (d) $S/I \cong \mathbb{Z}_2$, a field, therefore I is a maximal ideal of S.

4.26: (b) If we are to assume that $\phi(k) = k$ for all $k \in \mathbb{Z}$, then $\phi(\alpha)^2 = \phi(\alpha)\phi(\alpha) = \phi(\alpha^2) = \phi(a) = a$. This forces $\phi(\alpha) \in \{\alpha, -\alpha\}$, which forces

$$\phi(a + b\alpha) = \phi(a) + \phi(b\alpha) = \phi(a) + \phi(b)\phi(\alpha) = a + b\phi(\alpha) \in \{a + b\alpha, a - b\alpha\}.$$

So there are two automorphisms, the identity automorphism and the conjugation automorphism.