Homework 8 due Thursday, November 6:

Pages 87–88, problems 1.7, 1.8, 1.9, 1.16, 1.19 and pages 95–96, problems 2.2, 2.5.

Note: $F(\alpha)$ is notation for the field

$$F[\alpha] = \{f_0 + f_1\alpha + \cdots + f_n\alpha^n : n \geq 0, f_0, \ldots, f_n \in F\},$$

assuming that α is algebraic over F. More generally, $F(u_1, u_2, \cdots, u_n)$ is notation for the field

$$F[u_1][u_2]\cdots[u_n] = \{\sum_{i_1,i_2,\ldots,i_n} f_{i_1,i_2,\ldots,i_n} u_1^{i_1} u_2^{i_1} \cdots u_n^{i_n} : f_{i_1,i_2,\ldots,i_n} \in F\},$$

assuming that u_1 is algebraic over F, u_2 is algebraic over $F[u_1]$, u_3 is algebraic over $F[u_1][u_2]$, etc.

Hints:

1.7: In each case you must find a monic polynomial $p(x)$ satisfied by u with coefficients in the given field F and prove that $p(x)$ is irreducible in $F[x]$ using the usual techniques.

1.8: Use the Tower Theorem applied to the field extension $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}] \subseteq \mathbb{Q}[\sqrt{2}][\sqrt{3}]$.

1.9: In order to prove that $F[u_1, u_2, \ldots, u_j] = F[v_1, v_2, \ldots, v_k]$ it suffices to prove that $u_i \in F[v_1, v_2, \ldots, v_k]$ for $1 \leq i \leq j$ and that $v_i \in F[u_1, u_2, \ldots, u_j]$ for $1 \leq i \leq k$. You can sometimes prove that $F[u_1, u_2, \ldots, u_j] \neq F[v_1, v_2, \ldots, v_k]$ by showing that $[F[u_1, u_2, \ldots, u_j] : F] \neq [F[v_1, v_2, \ldots, v_k] : F]$, but equality of dimensions does not imply equality (or even isomorphism) of fields.

1.16: (a) is a consequence of Corollary [1.9], p. 78, which we proved in class. (b) Play with the fact that $\omega^3 = 2$, or alternatively with the fact that $\mathbb{Q}[\omega] \cong \mathbb{Q}[x]/(x^3 - 2)$.

1.19: See remarks for 1.16.

2.2: See remarks for 1.7.

2.5: This problem uses the same techniques that I used Tuesday, 10/28, to prove that $[\mathbb{Q}[2^{1/2}, 5^{1/3}] : \mathbb{Q}] = 6$.